GENERALIZED VECTOR MATRICES, HYPERCOMPLEX NUMBERS, AND DUAL NUMBERS : THEORY AND ITS APPLICATIONS TO SOME MATHEMATICAL PROBLEMS IN MECHANICS, ROBOTICS, AND OPTICS

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ชื่อวิทยานิพนธ์	เมทริกซ์เชิงเวกเตอร์ในนัยทั่วไป จำนวนเชิงซ้อนมิติเกิน และ จำนวนทวิภาค : ทฤษฎีและการประยุกต์เพื่อแก้ปัญหาทางคณิตศาสตร์ ในกลศาสตร์ วิทยาการหุ่นยนต์ และทัศนศาสตร์
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บทคัดย่อ

วัตถุประสงค์หลักของวิทยานิพนธ์นี้ คือ เพื่อศึกษาสมบัติเชิงพีชคณิต และ เรขาคณิต ของเมทริกซ์เชิงเวกเตอร์ในนัยทั่วไป (ซึ่งรวมทั้งส่วน สเกลาร์ และ เวกเตอร์ ไว้ในธาตุมูล) จำนวนเชิงซ้อนมิติเกิน และ จำนวนทวิภาค และเพื่อวิเคราะห์หาแนวทางประยุกต์สิ่งเหล่านี้ในการแก้ปัญหาทางคณิตศาสตร์ ใน สาขาวิทยาศาสตร์กายภาพ และ วิศวกรรมศาสตร์ ในเบื้องต้น ได้กล่าวถึงการพัฒนาระบบจำนวนแบบต่างๆ นับตั้งแต่ระบบจำนวนนับ จำนวนตรรกยะ จำนวนจริง จำนวนเชิงซ้อน และจำนวนเชิงซ้อนมิติเกิน ไปจนถึงจำนวนลักษณภาพเกินโดยทั่วไป นอกจากนี้ ยังกล่าวโดยย่อถึงการพัฒนา จำนวนพี-แอดิก จำนวนนอกมาตรฐาน (จำนวนจริงเกิน) และ จำนวนอติสัจยะ(จำนวนของคอนเวย์) การพิจารณาอัตลักษณ์ในทางพีชคณิต สำหรับโครงสร้างเชิงซ้อนมิติเกิน ของ จำนวนจตุรภาพ (ควอเทอร์เนียน) จำนวนอัฐภาพ(ออคโทเนียน)(หรือที่เรียกว่าจำนวนของเคลี่ย์) จำนวนทวิจตุรภาพ(ไบควอเทอร์เนียน)(ชนิดของแฮมิลตันและของคลิฟฟอร์ด) และ จำนวนทวิภาค เป็นต้น นั้น ได้นำมุ่งเข้าสู่การศึกษาเมทริกซ์เชิงเวกเตอร์ของซอร์น และต่อไปยังการขยายนัยทั่วไปของเมทริกซ์เชิงเวกเตอร์นี้ โดย อนาไจรอส จี. เฟลลูรีส โดย เหียว ชูล มวง และโดย ซูซูมุ โอคูโบ นอกจากนี้ ยังได้กล่าวถึงความเกี่ยวพันของโครงสร้างเหล่านี้กับพีชคณิตของ ลี และ ของ

ยอร์ดาน อันเป็นที่คุ้นเคยกันอย่างดีแล้ว โครงสร้างเชิงพีชคณิตของฟิสิกส์ที่สัมพันธ์กัน ก็ได้รับการพิจารณาด้วยในบางกรณี ได้แก่ พีชคณิตของ ปัวส์ซง และกลุ่มของ ไฮเซนแบร์ก รวมทั้งกลุ่มเชิงเมทริกซ์ที่มีความสำคัญในกลศาสตร์ตามแบบแผน และ เชิงควอนตัม เมทริกซ์ของ โจนส์ กับ มิลเลอร์ ในทัศนศาสตร์ทางด้าน พีชคณิต และ แคลคูลัส ของการเกิดขั้ว อีกทั้ง สปินเนอร์ กับ ทวิสเตอร์ ในทฤษฎีสัมพัทธภาพ และ ทฤษฎีควอนตัมเชิงสัมพัทธภาพ ด้วย ในการศึกษาจำนวนลักษณภาพเกินที่ขยายนัยทั่วไป (ของ ชาร์ลส์ มิวซีส) ซึ่งอาจรวมถึงกรณีต่างๆที่มีหน่วยหลัก ตัวหารศูนย์ ตัวบรรลุศูนย์ ตัวบรรลุอัตภาพ อย่างหลากหลาย นั้น ต้องการพื้นความรู้ทางพีชคณิตที่กว้างขึ้น จึงต้องศึกษาพีชคณิตแบบนอกการเปลี่ยนกลุ่ม และโครงสร้างเชิงพีชคณิตแบบอื่นๆ ซึ่งรวมสมบัติต่างๆเหล่านี้เข้าด้วยกัน แล้วแต่กรณี ได้แก่ การให้เกิดการเปลี่ยนกลุ่มของกำลัง หรือ การสลับ หรือ ภาวะอ่อนโยน การยอมรับโครงสร้างของ ลี หรือ ยอร์ดาน หรือ มาลเซฟ การให้บรรลุศูนย์ได้ และ การจัดลำดับขั้น ในปัจจุบันได้มีการนำสิ่งต่างๆ เหล่านี้ บางกรณีมาใช้ ใน ฟิสิกส์เชิงคณิตศาสตร์ และ ฟิสิกส์ทางทฤษฎี ตัวอย่างเช่น โครงสร้างของ ลี ที่มีการจัดลำดับขั้น ซึ่งมักรู้จักกันในนามของ อภิพีชคณิต และ กลุ่มอภิลักษณ์

สำหรับตัวอย่างที่น่าสนใจในการประยุกต์เพื่อแก้ปัญหาทางคณิตศาสตร์ ในกลศาสตร์ และ วิทยาการหุ่นยนต์ ที่ได้นำมาพิจารณา ก็ได้แก่ การนำจำนวนเชิงซ้อนมิติเกิน และ จำนวนทวิภาค มาใช้ในกลศาสตร์ของ ไจโรสโกป และในเรื่อง จลนศาสตร์ของแขนของหุ่นยนต์ ในด้านทัศนศาสตร์นั้น ได้เสนอการใช้เมทริกซ์เชิงเวกเตอร์ ในนัยทั่วไป ในการพัฒนาทฤษฎีเกี่ยวกับการเกิดขั้วบางส่วนของแสง ในกรณีที่มีการแปรปรวนของ วิถี และ ระดับขั้น ของการเกิดขั้ว นอกจากนี้ ได้นำเสนอวิธีการทางด้าน แผนภูมิรูปภาพทางคอมพิวเตอร์ เพื่อช่วยให้สามารถทำ "การคะนึงภาพ" ของเมทริกซ์เชิงเวกเตอร์ในนัยทั่วไป ได้ง่ายยิ่งขึ้น

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ABSTRACT

The main purpose of this thesis is to study the algebraic and geometrical properties of the generalized vector matrices (that include elements of both scalars and vectors), hypercomplex numbers, and dual numbers, and to investigate their applications to the solution of some mathematical problems in physical sciences and engineering. The development of "number systems" is first described, from natural numbers, through rational, real, complex, and hypercomplex numbers, to general *hypernumbers*. A brief survey of the p-adic, nonstandard (hyperreal), and surreal (Conway) numbers, is also given. The algebraic characterizations of the *hypercomplex* structures of the quaternions, octonions (or Cayley numbers), biquaternions (of Hamilton and of Clifford), dual numbers, etc., lead naturally to the study of the Zorn vector matrices, and their recent generalizations by Anargyros G. Fellouris, Hyo Chul Myung, and Susumu Okubo. The connections of these structures

with the familiar Lie and Jordan algebras are indicated. Some related algebraic structures of physics are discussed, including the Poisson algebras and the Heisenberg and important matrix groups in classical and quantum mechanics, the Jones and Mueller matrices of the algebra and calculus of polarization in optics, and the spinors and twistors in relativity and relativistic quantum theory. To provide an adequate basis for the study of the general *hypernumbers* (of Charles Muses), that may include a wide variety of units, zero-divisors, nilpotents and idempotents, a broader view of algebra is needed. This involves the study of more general nonassociative algebras, and other algebraic structures, with various combinations of such properties as power-associativity, alternativity, flexibility, Lie- or Jordan- or Malcev-admissibility, nilpotence, and grading. Some of these have recently been introduced into mathematical and theoretical physics, for example, the *graded* Lie structures that are now referred to as *superalgebras* and *supergroups*.

As examples of their applications to some interesting mathematical problems in mechanics and robotics, *hypercomplex* and *dual* numbers have been used in the dynamics of gyroscopes, and in the kinematics of spatial mechanisms and robot-arm manipulators. In optics, a suggestion is made for the use of generalized vector matrices in developing a theory of *partial* polarization of light with fluctuating mode and degree of polarization. A pictographical computer method has been introduced to provide an easy "visualization" of the generalized vector matrices.

TABLE OF CONTENTS

TABLE OF CONTENTS (Continued)

TABLE OF CONTENTS (Continued)

TABLE OF CONTENTS (Continued)

TABLE OF CONTENTS (Continued)

LIST OF TABLES

LIST OF FIGURES

LIST OF FIGURES (Continued)

LIST OF ABBREVIATIONS

NUMBERS

$\mathbb{N}$	Natural numbers
$\mathbb{R}$	Real numbers
$\mathbb{R}^+$	Positive real numbers
$\mathbb{C}$	Complex numbers
$\mathbb{H}$	Quaternions
$\mathbb{O}$	Octonions
i_n	i-type hypernumbers
ε_n	ε-type hypernumbers
$\hat{A}$	Dual number
ε	Dual unit

GROUPS, RINGS, FIELDS, AND MATRICES

$\langle G,*\rangle$	Group
$\langle R,+,\cdot\rangle$	Ring
$\mathbb{R}^*$	Hyperreal field
I	Identity matrix
A^{-1}	Inverse of matrix A
A^T	Transpose of matrix A
A^*	Complex conjugate of matrix A
A^+	Hermitian conjugate of matrix A
det A	Determminant of matrix A
Mn(K)	The set of all n×n matrices with elements from K
$\sigma_x,\sigma_y,\sigma_z$	2x2 Pauli matrices
$R(\theta)$	Rotation matrix
GL(n,K)	General linear groups
SL(n,K)	Special linear groups

LIST OF ABBREVIATIONS (Continued)

U(n)	Unitary groups
SU(n)	Special unitary groups
O(n)	Orthogonal groups
SO(n)	Special orthogonal groups
Sp(2n)	Symplectic groups

VECTORS, SPINORS, TWISTORS, AND VECTOR MATRICES

X	Vector X
$\underline{0}$	Zero vector
$\lvert X \rvert$	Magnitude of vector X
$\mathbb{V}$	Vector spaces
$T^{\alpha\bullet}{}_{\bullet}$	The notation for twistors
$\begin{bmatrix} a & X \\ Y & b \end{bmatrix}$	Vector matrix, where $a, b \in \mathbb{R}$, $X, Y \in \mathbb{V}$

OTHER

{ }	Set
$\in$	Membership (e.g., $a \in S$)
τ or TAU	Linear functional
[,]	Commutator
$[\ ,\ ]_{PB}$	Poisson bracket
$\subset$	Set inclusion
$\subsetneq$	Proper subset
$\leq$	Less than or equal to
$\geq$	Greater than or equal to
Σ	Summation

LIST OF ABBREVIATIONS (Continued)

$\dot{x}$	First derivative of x
$\ddot{x}$	Second derivative of x
N(X)	Norm of X
δ_{ij}	The Kronecker delta : $$\delta_{ij} = \begin{cases} 1 & i=j \\ 0 & i\neq j \end{cases}$$
ϵ_{ijk}	The Levi- Civita symbol : $$\epsilon_{ijk} = \begin{cases} 1 & \text{if } ijk = 123,\ 231 \text{ or } 312 \\ -1 & \text{if } ijk = 132,\ 213 \text{ or } 321 \\ 0 & \text{otherwise} \end{cases}$$
$\sigma(x)$	Sign function : $$\sigma(x) = \begin{cases} 0 & \text{if } x \text{ is even} \\ 1 & \text{if } x \text{ is odd} \end{cases}$$

CHAPTER I

INTRODUCTION

In mathematics, the great importance of *complex* numbers, *complex* analysis and *complex* differential geometry has long been recognized. In science and technology, in general, complex numbers, and functions of a complex variable are now being used for a wide range of theoretical and practical purposes, and some knowledge of them is thus required for the proper education of physical scientists and engineers. In physics, complex numbers and complex functions enter the basic formulations of relativity, quantum mechanics, and quantum field theory, in quite essential manners. Indeed, complex numbers, complex vector spaces, complex wave functions, as well as matrices, tensors, spinors and twistors, with complex elements, complex components, or complex parameters, can provide very effective descriptions of the natural order in the *relativistic* and *quantum* worlds. More familiarly, complex numbers and their functions have been used, for convenience, in the consideration of periodic structures, oscillatory behaviour and wave phenomena. This ranges from the theoretical treatment of the de Broglie waves in quantum mechanics to the practical calculations for the alternating-current circuits in electrical engineering.

The present chapter gives a mathematical and physical background and a general introduction to the other chapters of this thesis. It is divided into five sections, as follows : Section 1.1 gives a brief survey of the concept of "number", from natural numbers, via real, complex and hypercomplex numbers, to hypernumbers. p-Adic, nonstandard, and surreal (Conway) numbers are mentioned in Sections 1.2-1.4 , respectively. The developments in algebra, geometry and analysis arising from the use of hypercomplex numbers and hypernumbers are then presented in Section 1.5. Section 1.6 considers the applications of hypercomplex numbers, hypernumbers, and nonassociative algebras to problems in physics. In Section 1.7 the contents of this thesis are summarized.

Section 1.1 From Natural Numbers to Hypernumbers

Modern mathematics has significantly extended the ancient idea of "number". (See, e.g., [5],[19],[22].) The widely recognized *number systems* now include the following: $\mathbb{N} \subset \mathbb{Z} \subset \mathbb{Q} \subset \mathbb{R} \subset \mathbb{C} \subset \mathbb{H} \subset \mathbb{O}$, where $\mathbb{N}$ is the set of natural numbers, $\mathbb{Z}$ the set of integers, $\mathbb{Q}$ the set of rational numbers, $\mathbb{R}$ the set of real numbers, $\mathbb{C}$ the set of complex numbers, $\mathbb{H}$ the set of quaternions, and $\mathbb{O}$ the set of octonions or Cayley numbers. In particular, the familiar concept of a "complex" number, $\zeta = (x + iy)$ (where x and y are real numbers), with i representing $\sqrt{-1}$, has been generalized into that of a "hypercomplex" number, which includes the case of the quaternions (see Chapter II, Section 6) and that of the octonions (see Chapter III, Section 1), etc., where the meaning of $\sqrt{-1}$ has been considerably extended.

The idea of "hypernumbers" beyond that of the *hypercomplex* cases is also becoming an important part of mathematics. The motivation to study some of them has been provided by attempts to find suitable *number systems* for the mathematical formulation of some fundamental physical theories. In contrast to the *hypercomplex* systems of quaternions and octonions, these *hypernumbers* may include generalized idempotents, zero-divisors, nilpotents, and a wide variety of noncommutative or nonassociative units other than those of the quaternions and the octonions. Hypernumbers give some new domains of algebra, geometry, and function theory. There are good reasons to expect that the mathematical results of these domains will soon find important applications in the physical sciences and in engineering.

In 1843 the Irish mathematician Sir William Rowan Hamilton (1805-1865) first developed the concept of the "quaternions" as a generalization of the familiar complex numbers.([2],[3],[5],[6],[19],[22],[32],[68],[69],[73],[74]) A *quaternion* is the quadruple (x_1,x_2,x_3,x_4) which has the basis (1,i,j,k) with the following multiplication rules:

$$i^2 = j^2 = k^2 = -1 ,$$
$$ij = -ji = k , \qquad \text{----- (1.1)}$$
$$jk = -kj = i , \quad ki = -ik = j .$$

Let $e_1 = 1$, $e_2 = i$, $e_3 = j$, $e_4 = k$. Then the multiplication table of the quaternion units is as follows :

Table 1.1 The multiplication table of the quaternion units

	e_1	e_2	e_3	e_4
e_1	e_1	e_2	e_3	e_4
e_2	e_2	$-e_1$	e_4	$-e_3$
e_3	e_3	$-e_4$	$-e_1$	e_2
e_4	e_4	e_3	$-e_2$	$-e_1$

The quaternions form a *division ring* (see the definition in Chapter II, Section 1), and the multiplication is, in general, not commutative. The study of quaternions led to many other algebras, such as quadrinomes, tetrads, quines, pluquaternions, nonions, and tettarions. An important extension of the structure of quaternions is the systems of *biquaternions* of Hamilton and William Kingdon Clifford (1845-1879), which form 8-dimensional algebras. Hamilton's biquaternions obey the multiplication rules of the ordinary quaternions, but the ground field is now the set of all *complex* numbers. Clifford's biquaternions can be obtained from the ordinary quaternions by an "adjunction" of an element E which commutes with all the quaternions and satisfies $E^2 = I$. Both systems of biquaternions contain zero-divisors. If we consider an analogue of the four-square formula in an 8-dimensional algebra, however, we can construct 8-dimensional algebras without zero-divisors, such as that of the octonions (see Chapter III, Section 1).

In 1843, within a few months of Hamilton's invention of the quaternions, John T. Graves discovered a system of 7 imaginary units (including the three units of quaternions) in addition to a real unit, and gave the name "octads". He immediately communicated this discovery to Hamilton, who later (in 1848) gave the new name "octaves".

In 1845, Arthur Cayley (1821-1895) rediscovered the octaves. He had carefully studied the early publications of Hamilton on the quaternions; but without knowing anything about the important result that Graves had privately communicated to Hamilton in December 1843, he reported his independent "discovery" and invention in a postscript to his paper on a generalization of doubly-periodic functions. Octaves or octads are now most often referred to as "octonions" or "Cayley numbers". Graves and Cayley gave the idea of a system with 7 imaginary units which can be constructed as shown in Fig. 3.1 and Table 3.1A, that is :

$$e_A^2 = -1 \ , \quad e_A e_B = \varepsilon_{ABC} e_C - \delta_{AB} \ , \qquad \text{-----(1.2)}$$

where δ_{AB} is the Kronecker delta :

$$\delta_{AB} = \begin{cases} 0 , & A \neq B \\ 1 , & A = B \end{cases}$$

and ε_{ABC} is the totally antisymmetric Levi-Civita symbol :

$$\varepsilon_{BAC} = \varepsilon_{ACB} = \varepsilon_{CBA} = -1$$

and $$\varepsilon_{ABC} = \varepsilon_{BCA} = \varepsilon_{CAB} = +1$$

for ABC $\in$ {123, 147, 165, 246, 257, 354, 367}.

The multiplication in this system is neither commutative nor associative :

$$e_A e_B = -e_B e_A \quad ,$$

and $$(e_A e_B) e_C = -e_A (e_B e_C) \quad . \qquad \text{-----(1.3)}$$

Both Graves and Cayley hoped to extend the system of complex units to higher cases with 2^n(n>3) complex units (including one real unit). Graves made an attempt with 16 units, but without success. Cayley considered it plausible that such a system would exist if 2^n-1 is a prime number. We now know, however, that the systems with 1, 2, 4 or 8 units are the only possible cases, if divisors of zero are excluded and if the distributive law is to remain valid.

Recently, mathematicians and physicists have successfully extended the systems of *higher-dimensional numbers* to 16 dimensions, and call them *hypernumbers*.

The arithmetic of such hypernumbers, however, has divisors of zero and are not associative. In 1970, Charles Muses, who is the author of studies on higher algebras, modern physics, and cybernetics, extended Clifford's work on the biquaternions, and introduced hypernumbers of type ε and i. The relationship between ε numbers and i numbers gives a very important form of $\sqrt{-1}$ which plays a basic role in Dirac's quantum physics. After three years, Muses introduced the bimatrix, defined as a pair of separated matrix domains connected by certain rules (see Chapter III, Section 3), and soon after used bimatrices in solving Cayley arithmetic and algebra.

In 1873, W. K. Clifford introduced *dual numbers* of the form $\hat{A} = A + \varepsilon A_o$ (the analogues of complex numbers) where A and A_o are real numbers and $\varepsilon^2 = 0$. Now, we find that the dual operator ε is nilpotent of second order, i.e., it is a *zero-divisor*. Also, it is known that ε can be generated from two hypernumbers: $k(i_n \pm \varepsilon_m)$, where k is a real number, i_n and ε_m are hypernumbers of the two types (see Chapter III, Section 3), and $n \neq m$. Thus, dual numbers are also hypernumbers.

Hypernumbers are at the very core of mathematics, and they provide a powerful approach to new frontiers. Each kind of hypernumber has its own arithmetic, and, usually, its own geometry as well. However, in this thesis, we will consider only those hypernumbers that seem to promise important and interesting applications to problems in contemporary physics and engineering.

Section 1.2 p-Adic Numbers

In 1913, the German mathematician Kurt Hensel (1861-1941) introduced the theory of p-adic numbers by modelling them on the power series of complex function theory. In the other way, we can determine them as a natural completion of the field of rationals, just as the reals are the completion of the rationals. The p-adic number is a *non-Archimedean* valuation (see the definition in Section 1.2.2), and can be thought of as the completion when the absolute value is replaced by a p-adic valuation. ([13],[22],[37],[48],[72])

1.2.1 Number as function

We will now consider numbers as functions. We start from the polynomial function

$$f(z) = a_o + a_1 z + \dots + a_n z^n \qquad \text{----- (1.2.1)}$$

where the a_i are complex coefficients, or $a_i \in \mathbb{C}$, z is a complex variable. Then, f(z) is a function on the complex plane. In the case of the polynomials $f(x) \in \mathbb{C}(z)$, the higher derivatives at the point $z = a$ are given by the coefficients of the expansion

$$f(z) = a_o + a_1(z-a) + \dots + a_n(z-a)^n \qquad \text{.----- (1.2.2)}$$

We are thus led to the concept of p-adic numbers by observing that every rational number $f \in \mathbb{Q}$ can be given an analogous expansion with respect to every prime element p of integer. Every positive integer f can be written as prime fraction

$$f = \sum_{\nu=-m}^{\infty} a_\nu p^\nu \qquad \text{----- (1.2.3)}$$

in which the coefficients a_i lie in $\{0,1,...,p-1\}$, that is, in a fixed representative system of the "field of value".

DEFINITION : Let p be a fixed prime. A p-adic number is a formal infinite series

$$a_{-m}p^{-m} + ... + a_{-1}p^{-1} + a_0 + a_1 p + a_2 p^2 + ... ,$$

in which $a_i \in \{0,1,...,p-1\}$. The p-adic integers are the series

$$a_0 + a_1 p + a_2 p^2 + ...$$

The complete set of all p-adic numbers is denoted by $\mathbf{Q}_p$, and that of all p-adic integers by $\mathbf{Z}_p$. Addition and multiplication can be defined for p-adic numbers, whereby $\mathbf{Z}_p$ becomes a ring whose quotient field is $\mathbf{Q}_p$.

1.2.2 Pseudo-valuations and valuations

Let K be a commutative ring with the unit (identity element) 1.

DEFINITION : A map W from K into the set of non-negative real number is called a *pseudo-valuation* if it has the following properties :

$W(0) = 0$, $W(a) > 0$ if $a \in K$ is not 0 . ----- (1.2.6)

$W(a \pm b) \leq W(a) + W(b)$ for all $a,b \in K$. ----- (1.2.7)

$W(ab) \leq W(a)W(b)$ for all $a,b \in K$. ----- (1.2.8)

If (1.2.7) holds in the stronger form

$W(a \pm b) \leq \max(W(a),W(b))$ for all $a,b \in K$,----- (1.2.9)

then W is said to be a *non-Archimedean pseudo-valuation;* otherwise it is said to be *Archimedean* . If further (1.2.8) holds in the strengthened form as an equation,

$W(ab) = W(a)W(b)$ for all $a,b \in k$, ----- (1.2.10)

then W is called a *valuation*. Each of the properties (1.2.7) , (1.2.8) , (1.2.9) , and (1.2.10) may be applied repeatedly and leads, for every positive integer n and any n elements $a_1, a_2, \ldots, a_n$ of K , to

$$W\left[\sum_{k=1}^{n} a_k\right] \begin{cases} \leq \sum_{k=1}^{n} W(a_k) \\ \leq \max_k W(a_k), & \text{if } W \text{ is non-Archimedean,} \end{cases}$$

and

$$W\left[\sum_{k=1}^{n} a_k\right] \begin{cases} \leq \prod_{k=1}^{n} W(a_k) \\ = \prod_{k=1}^{n} W(a_k), & \text{if } W \text{ is a valuation.} \end{cases}$$

Section 1.3 Nonstandard Numbers

In 1960, Abraham Robinson (1918-1974) extended the field $\mathbb{R}$ of real numbers to a field $^*\mathbb{R}$ of *hyperreal* numbers in which there are both *infinitely small* and *infinitely large* "numbers". This has led to a *Leibnizian* approach to *infinitesimal calculus* that uses *infinitesimal* numbers directly and explicitly. A *derivative* is thus the quotient of two infinitesimals; an *integral* now means the sum of infinitely many infinitesimals. The development of a *theory of infinitesimals* requires a decision as to which properties of $\mathbb{R}$ shall remain true of the extension. The class of such properties must be broad enough to permit computing with infinitesimals as though they were real, but not so broad that they force $^*\mathbb{R}$ to be just $\mathbb{R}$. This approach to infinitesimal calculus was the beginning of a new branch of mathematics, *nonstandard analysis*, that covers mathematical operations carried out within the *hyperreal* field $^*\mathbb{R}$. ([3],[18],[22],[40],[47],[65],[70])

The field $^*\mathbb{R}$ is an ordered overfield of $\mathbb{R}$, in which there are elements a with the property that $r<a$ for all $r\in\mathbb{R}$; in other words, elements a which are infinitely large. Clearly, any element $1/a$ is then infinitely small; it lies between 0 and all positive $\varepsilon\in\mathbb{R}$. Thus we can define

$$S = \text{set of non-infinitely large elements of } {}^*\mathbb{R}$$
$$= \{ s \in {}^*\mathbb{R} \mid |s| \leq r \quad \text{for an} \quad r \in \mathbb{R} \} \quad .$$

One can quickly verify that S is a subring of $^*\mathbb{R}$.

In S we find

$$J = \text{set of infinitesimally small elements}$$
$$= \{ t \in {}^*\mathbb{R} \mid |t| \le \varepsilon \quad \text{for all} \quad \varepsilon \in \mathbb{R}^+ \} \quad ,$$

where $\mathbb{R}^+$ is the positive real numbers. On using st for "standard", the homomorphism st : $S \longrightarrow S/J$ is called the transition to the standard part of $s \in S$, and the coset $s+J$ is called the *monad* of s. These elements of ${}^*\mathbb{R}$, that do not lie in S, have no standard part, even though one defines a monad for them. Now the fact that the standard part of $s \in S$ is the coset $s+J$ (the monad of s) does not help us very much, because we really wanted a number in $\mathbb{R}$. Therefore, we now show that in each monad, there lies exactly one real number, and this yields $S/J = \mathbb{R}$. This will justify the description of the *real* number belonging to the monad of s as the *standard* part of s. Elements $x, y \in {}^*\mathbb{R}$ are not said to be neighbouring elements or neighbours, if $x-y>0$ in $\mathbb{R}$, x and y are not infinitesimally neighbouring in S, therefore $stx \neq sty$ in S/J.

Nonstandard analysis provides natural mathematical models of many situations where one's intuition involves infinite or infinitesimal quantities which are *non-Archimedean* ordered. For example, in probability theory, where one thinks of the probability of an event as the (infinite) number of favourable cases divided by the number of all cases, and in physics, where one thinks of a quantum field with infinite fluctuations in infinitesimal regions. The growing number of applications of *nonstandard* methods is likely to convince more applied mathematicians of the value of learning and teaching these methods.

Section 1.4 Surreal (Conway) Numbers

A number becomes *infinite* in extent in many different ways. One case into which mathematicians has delved more deeply is that of the *surreal* numbers, or Conway numbers. ([1],[15],[22],[42]) In dealing with logic and set theory, John Horton Conway introduced a definition of a large ordered number field. The elements of this field can be interpreted as "games", now referred to as "Conway games". This theory has been applied to geometry, and involves the concept of a "winning strategy".

1.4.1 Ordered field

A field K is called an *ordered field* if there is given a total order in K such that $a>b$ implies $a+c > b+c$ for all c, and $a>b$, $c>0$ implies $ac > bc$. The characteristic of an ordered field is always 0. If, for any two positive elements a, b of K, there exists a natural number n such that $na > b$, then we call K an *Archimedean ordered field* . Two ordered fields are called *similarly isomorphic* if there exists an isomorphism between them under which positive elements are always mapped to positive elements. The rational number field and the real number field are examples of Archimedean ordered fields, while every Archimedean ordered field is similarly isomorphic to a subfield of the real number field.

1.4.2 Dedekind's definition

Dedekind proposed the abstract definition of pure *numbers*. Suppose the rational numbers are divided in any way into two classes, L, R, say, such that

i) all the members of L are less than all the numbers of R;

ii) L contains at least one rational number;

iii) all the rational numbers belong either to L or to R.

Any such division is called a *section* of the rational numbers. Then there are two different kinds of section possible :-

a) Either one number in L is greater than all the other numbers in L or one number in R is less than all the other numbers in R ("or" here excludes "and").

b) Or no number in L is greater than all the other numbers in L and no number in R is less than all the other numbers in R.

1.4.3 Conway games

Let L and R be subclasses of a totally-ordered class T. We write L<R if $x^L \in L$, and $x^R \in R$ implies $x^L < x^R$. Note that $\phi<R$ and $L<\phi$ for all L and R. Conway constructs the elements of No (positive infinite number) as follows : If L and R are two subsets of No with L<R, then there exists a number $\{L|R\} = x \in$ No. All elements of No are constructed in this way.

In general, if $x = \{L|R\}$, then x^L will denote a typical element of L, and x^R a typical element of R. Conway defines

$$x \geq y \text{ iff } x^R > y \text{ and } x > y^L \text{ for all } x^R \text{ and } y^L.$$

It is well to note that equality between these numbers is an equivalence relation, namely

$$x = y \quad \text{iff} \quad x \geq y \quad \text{and} \quad y \geq x \quad ,$$

and note the following : $\{L|R\} = \{L'|R'\}$ iff $L=L'$ and $R=R'$. Conway shows that No is a proper class which is also a real-closed field. Note if $x = \{L|R\}$, then $L<\{x\}<R$; thus, whenever a gap exists in the numbers defined thus far, an element is created to fill that gap. This process of creation occurs only when L or R is a proper class of numbers. We can sum up the Conway game as the simple game below.

Conway games are defined inductively. If x and y are any two sets of games, then there is a game (x,y). All Conway games are constructed in this way.

Numbers constitute a subclass, and they are also defined inductively : if x^L and x^R are any two sets of numbers, and no number of x^L is $\geq$ any member of x^R, then there is a number (x^L,x^R). All numbers are constructed in this way.

The empty set (ϕ,ϕ) serves to define the number created on day 0 : $(\phi,\phi) = 0$. On day 1, the numbers

$(\phi,\{0\}) = -1$, $(\{0\},\phi) = 1$, and the game $(\{0\},\{0\}) = *$ are created; and on day 2 we get, among other numbers, $(\{0\},\{1\}) = \frac{1}{2}$, and among other games, $(\{1\},\{-1\}) = \pm 1$, $(\{0\},*) = \uparrow$. (See the "partizan game-graphs" in Fig.1.4, in which vertices are game positions, and edges slanted in south-westerly direction denote moves of Left; in south-easterly direction moves of Right.) In particular, it can be seen that the following sets are Conway games:

$$
\begin{aligned}
1 &= (\{0\},\phi) \\
2 &= (\{0,1\},\phi) \\
&\vdots \\
n+1 &= (\{0,\ldots,n\},\phi)
\end{aligned}
$$

Every Conway game is a pair of sets, and all Conway numbers are called ordinal numbers.

Definition : An *ordinal number* is a well-ordered set in which each element is equal to the set of all its predecessors. In other words, a set X, well-ordered by R, is an ordinal number if for each $x \in X$

$$x = \{ y \in X \mid yRx \ \& \ y \neq x \}.$$

The well-ordering theorem implies that $\geq$ is a total ordering. $m>n$ means that $m \geq n$ and $m \neq n$.

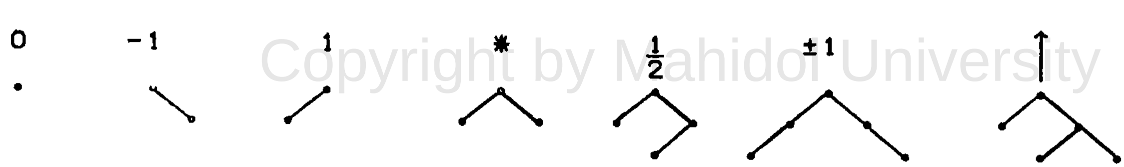


Fig.1.4 A few partizan games

Section 1.5 Algebra, Geometry, and Analysis

The foregoing sections outline the long development of the idea of "number". In this section, the connections of the hypercomplex number systems (in particular, the quaternions and octonions), and of the hypernumbers, with algebra, geometry and analysis, will be briefly considered.

In 1844, Hermann Grassmann (1809-1877) developed his *theory of extension* for algebraic and geometric systems in spaces of more than three dimensions. Later on Clifford studied Grassmann algebras, and developed a *geometry of motion*, for the study of which he generalized Hamilton's quaternions into the so-called *biquaternions* .

In 1870, (Marius) Sophus Lie (1842-1899) gave the theory of all continuous groups, embracing all the theories of invariants hitherto noted. Lie's theory, the original theory of continuous groups, has been very successful in their applications to the differential equations of mechanics and mathematical physics. Lie groups and Lie algebras satisfy the anticommutative laws and have, in general, nontrivial commutators. Some special cases lead naturally to the theory of octaves.

In 1877, (Ferdinand) Georg Frobenius (1849-1917) established an interesting connection between hypercomplex number systems and *bilinear forms*. In 1884, Jules Henri Poincaré (1854-1912) discovered a further connection of these with the continuous groups of Lie .

In 1891, Georg Scheffers (1866-1945), who continued the lineage from Lie, enunciated that Lie's theory of (finite) continuous groups also contains the theory of hypercomplex number systems. After seven years, Élie Cartan (1869-1951) applied the Lie theory to obtain a classification of the hypercomplex number systems. The theory of these continuous groups also afforded principles for the classification of linear associative algebras.

The modern developments of algebra and number theory that have followed the generalization of the concept of "number" have been the results of an extended study by many mathematicians. One important mathematician, Adolf Hurwitz (1859-1919), extended and simplified the algebraic theory of forms, by considering, in particular, the invariants under the infinitesimal transformations of a certain Lie group. In 1898, he presented this development in a more concise form, which deepened into a very general law of the formation of independent categories in successively higher algebras. Hurwitz's theorem can be stated as follows : A necessary and sufficient condition for the existence of a nondegenerate quadratic form N(x) satisfying N(x)N(y) = N(z), where z is bilinear in x and y, (i.e., for permitting composition on a algebra A), is that A is one of the following : F_1 ; $F \oplus F$; a separable quadratic field $\mathbb{S}$ over F ; a complex $\mathbb{C}$ over F, a quaternion algebra $\mathbb{H}$ over F ; a Cayley algebra $\mathbb{O}$ over F. Hence the possible dimensions for A are 1, 2, 4, 8. An extension of the 8-square formula to a

16-square formula, and further on, is not possible.

In 1914, Leonard Eugene Dickson (1874-1954) constructed division algebras in n fundamental units with coefficients in any field F. In 1933, Max August Zorn (1906-) studied alternative rings, and introduced *vector matrices* which include elements of scalars and three-dimensional vectors in the same system. Zorn's vector matrices describe the geometry of an 8-dimensional space. Zorn's work also stimulated other mathematicians to further study the connections between geometrical axioms and algebraic rules. The addition of vector matrices is defined in an obvious way; but their multiplication is, in general, not associative (see Chapter III, Section 1).

In 1948, Abraham Adrian Albert (1905-1972) introduced Lie-admissible algebras, which are represented by the octonions, as can be seen from the following definitions : Given an algebra A over a field of characteristic $\neq 2$ with multiplication denoted by xy, we associate an anticommutative algebra A^- with multiplication $[x,y] = xy-yx$ defined on the vector space A. Then, A is termed *Lie-admissible* or *Malcev-admissible* if A^- is a Lie or a Malcev algebra. Lie-admissible algebras can be extended to Malcev-admissible algebras by using the standard theory of Lie and Malcev algebras which satisfy the anticommutative law and Malcev identity (see Chapter III, Section 4, Eq.(3.4.3)).

Section 1.6 Applications of Hypercomplex Numbers and Nonassociative Algebras in Physics

Murat Günaydin and Feza Gürsey (1921-) in a paper of 1973 [30] attempted to generalize quantum mechanics by using the octonions through the associativity condition for physical observables. They related the octonions to the split octonions by using a split basis, and thus obtained a formalism that exhibits the quark structure and its SU(3) content in charge space. They tried to construct a unified theory that would include gravity, starting from string theories in higher dimensions, by exploiting the relationship between the exceptional Lie algebras and the octonions.

In 1979, Susumu Okubo (1930-) attempted to use flexible Lie-admissible algebras A to generalize the Heisenberg equation whose solutions require the underlying algebra A to be power-associative. He admitted that he did not know how to introduce a Hilbert space into his analysis, but did speculate on the existence of quantum mechanical states via the notion of positive linear functionals. After two years, Okubo gave examples to point out the nonuniqueness of the Lagrangian, and suggested possible applications of flexible Lie-admissible algebras to quantum mechanics.

Section 1.7 Summary of Contents

This thesis begins with a background in mathematics and physics. We survey and study algebraic and geometrical properties of the *vector matrices* that include simultaneously elements which are vectors and scalars. The summary of the contents is as follows: Chapter II contains the definitions of classical groups, some basic concepts of Lie algebras, Lie groups, nonassociative algebras, nilpotent algebras, Jordan algebras, and alternative algebras. The structure of Poisson algebras, Heisenberg groups and Hamiltonian systems are next considered. The algebra and calculus of polarization are then introduced, with the linear, circular and elliptical polarizations being represented by vectors and matrices, in particular, by Jones vectors, and Jones and Mueller matrices. The representations of spin states, which can be transformed to holomorphic functions of spinors and twistors, are given in the last part of this chapter. In Chapter III, generalized vector matrices, bimatrices and octonions are considered. The octonion or Cayley algebra in the split basis and Zorn's vector matrices are first introduced. We next investigate the graded Lie admissibility of the vector matrix algebra. The concept of a bimatrix, defined as a pair of separated matrix domains connected by certain rules, is then introduced. A study of power associative products on octonions, as determined by third and fourth power laws, and flexibility as defined on the vector matrices, leads naturally to the consideration of Malcev-admissible algebras. Chapter IV is concerned with the representation of vector matrices by a pictographical computer method. The applications of vector matrices, hypercomplex numbers and dual numbers in physics and engineering, such as to the problems of multi-rigid-body-systems in mechanics, of the kinematic analysis, computation and control of motions in robotics, and of polarization in optics, are explained in Chapter V. The last chapter discusses further problems in the applications of generalized vector matrices, bimatrices, hypercomplex numbers and hypernumbers, to physics and engineering, by using the fundamental concepts given in the foregoing chapters. This then concludes with a survey of the range of problems for which the use of octonions and vector matrices is particularly useful and convenient.

CHAPTER II

SOME ALGEBRAIC STRUCTURES OF PHYSICS

In this chapter, some important algebraic structures that have already been used in physics will be considered. This should provide an adequate mathematical background for understanding the later chapters of this thesis. There are six sections altogether. Section 1 gives a brief summary of the properties of groups, matrices, and some important matrix groups. Section 2 describes Lie algebras, their properties, and the associated Lie groups. Section 3 presents the set of axioms for linear, nonassociative, and nilpotent algebras. As specific cases, Jordan and alternative algebras will be considered. In Section 4, Poisson algebras, the Heisenberg groups and the Lie algebraic structure of the Hamiltonian systems will then be discussed. The fifth section will be on the subject of the algebra and calculus of polarization of electromagnetic radiation. The cases of linear, circular and elliptical polarizations, and the use of the Stokes parameters and of the Jones vectors and Jones matrices to describe polarization will be considered. The last section will give a brief introduction to the physical applications of spinors and twistors. The proofs of the propositions in this chapter are, in general, straightforward, and have been omitted. They are given in [16], [27], and [77] .

Section 2.1 Matrix Groups

The main purpose of this section is to explain the different types of classical groups or matrix groups and to enable the reader to differentiate the various symbols used for matrix groups. (For further details, see [16],[17],[27],[77].)

2.1.1 Groups, rings, and fields

DEFINITION : A *binary operation* * on a set is a rule that assigns to each ordered pair of elements of the set some element of the set.

DEFINITION : A binary operation * on a set S is *commutative* if and only if $a*b = b*a$ for all $a,b \in S$. The operation * is *associative* if and only if

$(a*b)*c = a*(b*c)$ for all $a,b,c \in S$.

DEFINITION : A *group* $\langle G,*\rangle$ is a set G, along with a binary operation * on G and the following required properties of the operation :

(1) The binary operation * is associative. This means that for any $a, b, c \in G$ we have $(a*b)*c = a*(b*c)$. (Note that the closure of G with respect to * is implied.)

(2) There exists an identity element I of G. This means that for any $a \in G$ we have $Ia = aI = a$.

(3) Inverses exist. This means that for any $a \in G$ there is an element $a^{-1} \in G$ such that $aa^{-1} = a^{-1}a = I$.

DEFINITION : A group G is *abelian* if its binary operation $*$ is commutative.

DEFINITION : A *ring* is an algebraic system consisting of a set R and two binary operations, viz. addition (+) and multiplication (•), on R such that for all $a,b,c \in R$:

(A1) $(a+b)+c = a+(b+c)$.

(A2) There exists $0 \in R$ such that for all $a \in R$, $0+a = a+0 = a$.

(A3) Given $a \in R$, there exists $-a \in R$ such that $a+(-a) = (-a)+a = 0$.

(A4) $a+b = b+a$.

(M1) $(ab)c = a(bc)$.

(D) $a(b+c) = ab + ac$, $(b+c)a = ba + ca$.

DEFINITION : If R is a ring, $u \in R$ is a *unit* if there exists some $a \in R$ such that $au = ua = 1$, i.e., if it has a multiplicative inverse.

DEFINITION : A ring R with a multiplicative identity 1 such that $a \cdot 1 = 1 \cdot a = a$, for all $a \in R$, is a *ring with unity*. The identity in a ring is called a *unity*.

DEFINITION : If R is a ring with an associative multiplication and $U \subset R$ is the set of units u in R, then U is a *group* under multiplication.

DEFINITION : Let R be a ring with unity. An element u in R is a *unit* of R if it has a multiplicative inverse in R. If every non-zero element of R is a unit, then R is a *skew field* or *division ring*.

DEFINITION : If R is a division ring with the multiplication satisfying the commutative law ab = ba , then R is a *field*. A field is a *commutative division ring*.

2.1.2 Matrix properties

The properties of a number of special matrices are now tabulated : (See the definitions in [83], pp.280-317 .)

Table 2.1 Matrix properties

I_n = $n \times n$ unit matrix, abbreviated I .

$a \cdot I$ = constant matrix. $\mathrm{Tr}\, A = \sum_i A_{ii}$.

matrix	notation	Definition	
Transpose	A^T	$(A^T)_{ij} = A_{ji}$	
Conjugate	A^*	$(A^*)_{ij} = (A_{ij})^*$	
Adjoint	A^+	$A^+ = (A^T)^*$	a
Reciprocal	A^{-1}	$A^{-1}A = I$	b
Real (Re)		$A^* = A$	
Imaginary (Im)		$A^* = -A$	
Orthogonal		$A^TA = I$	b
Unitary		$A^+A = I$	b
Symmetric		$A^T = A$	
Skew-symmetric		$A^T = -A$	
Hermitian		$A^+ = A$	
Skew-Hermitian		$A^+ = -A$	
Real orthogonal		$A^TA = I$ and	
		$A^* = A \longrightarrow A$ unitary	c
Real symmetric		$A^T = A$ and	
		$A^* = A \longrightarrow A$ Hermitian	c
Real skew-symmetric		$A^T = -A$ and	
		$A^* = A \longrightarrow A$ skew-Hermitian	c
Non-singular		A^{-1} exists (det $A \neq 0$)	
Traceless		$\mathrm{Tr}\ A = 0$	

[a] The symbols ^{T} and * can be interchanged.

[b] The matrices on the left can be interchanged if A is non-singular.

[c] The converse entailment is not valid.

The groups involving regular matrices (see the definition in [77]) may be finite or infinite, be discrete or continuous, and have real ($\mathbb{R}$) or complex ($\mathbb{C}$) elements. The variables in the real space $\mathbb{R}^n$ are designated as $\mathbf{x} \cdot (x_1,\ldots,x_n)$, and in the complex space $\mathbb{C}^n$ as $\mathbf{z} = (z_1,\ldots,z_n)$. A regular matrix of degree n acting in $\mathbb{R}^n$ or $\mathbb{C}^n$ will produce a transformation $\mathbf{x} \longrightarrow \mathbf{x}'$ or $\mathbf{z} \longrightarrow \mathbf{z}'$. In problems in physics we are frequently interested in classes of transformations that leave invariant some functional form of $\mathbf{x}$ or $\mathbf{z}$. For example, in an isotropic three-dimensional Euclidean space we may wish to consider transformations that hold the form $x_1^2 + x_2^2 + x_3^2$ as an invariant, or in a 4-dimensional Lorentzian space the form $x_1^2 + x_2^2 + x_3^2 - x_4^2$.

2.1.3 The general linear groups GL(n,K)

DEFINITION : The group of units in the algebra $M_n(\mathbb{R})$ is denoted by $GL(n,\mathbb{R})$, in $M_n(\mathbb{C})$ by $GL(n,\mathbb{C})$, in $M_n(\mathbb{H})$ by $GL(n,\mathbb{H})$ and in $M_n(\mathbb{O})$ by $GL(n,\mathbb{O})$. These are the general linear groups and form linear algebras.

We know that

$$GL(n,K) = \{ A \in M_n(K) \mid \det A \neq 0 \}$$

where $K = \{ \mathbb{R},\mathbb{C},\mathbb{H},\mathbb{O} \}$,

$GL(n,K) \subset M_n(K)$,

and $GL(n,\mathbb{R}) \subset GL(n,\mathbb{C}) \subset GL(n,\mathbb{H}) \subset GL(n,\mathbb{O})$.

2.1.4 The special linear groups $SL(n,K)$

The *special linear group* $SL(n,K)$ is a general linear group of matrices with determinant = +1, and is characterized by $i_K(n^2-1)$ parameters, where $K = \{ \mathbb{R},\mathbb{C},\mathbb{H},\mathbb{O} \}$, and having $i_{\mathbb{R}} = 1$, $i_{\mathbb{C}} = 2$, $i_{\mathbb{H}} = 4$ and $i_{\mathbb{O}} = 8$. Clearly,

$$SL(n,K) \subset GL(n,K) \subset M_n(K) ,$$

and

$$SL(n,\mathbb{R}) \subset SL(n,\mathbb{C}) \subset SL(n,\mathbb{H}) \subset SL(n,\mathbb{O}) .$$

2.1.5 The unitary groups $U(n)$

In $GL(n,K)$, for $K=\mathbb{C}$, we write it as $U(n)$ and call it the *unitary group* , if the unitary matrices A of degree n form the elements of the n^2-parameter unitary group $U(n)$ that leaves the Hermitian form $\sum_{i=1}^{n} z_i z_i^*$ invariant. Since the unitarity of the matrices A requires that $A^+A = I$ the range of the matrix elements a_{ij} is restricted by the requirement that

$$\sum_t a_{it} a_{tj}^* = \delta_{ij} \quad ; \quad \text{hence,} \quad |a_{ij}|^2 < 1.$$

Thus, in this case the domain of the n^2 parameters is bounded, so we get $U(n) \subset GL(n,\mathbb{C})$.

2.1.6 The special unitary groups $SU(n)$

If we focus on unitary matrices of determinant +1, we will obtain the (n^2-1)-parameter *special unitary group* or *unitary unimodular group* $SU(n)$, and we will find that

$$SU(n) = U(n) \cap SL(n,\mathbb{C}).$$

2.1.7 The orthogonal groups O(n)

In GL(n,K), for $K=\mathbb{R}$, we write $O(n,\mathbb{R})$ as O(n) and call it the *orthogonal group* when $A^tA = I$ and we have $\det A = \pm 1$. The set of real orthogonal matrices of degree n forms the n(n-1)-parameter group $O(n) \subset GL(n,\mathbb{R})$.

2.1.8 The special orthogonal groups SO(n)

The set of real orthogonal matrices of degree n forms the $\frac{1}{2}$ n(n-1)-parameter *real orthogonal group* $O(n,\mathbb{R})$, while the set of real orthogonal matrices of determinant +1 forms the *real special orthogonal group* $SO(n,\mathbb{R})$. The real special orthogonal matrices leave invariant the real quadratic form $\sum_{i=1}^{n} x_i^2$ We have

$$SO(n) = \{ A \in U(n) \mid \det A = 1 \} ,$$

and $$SO(n) = SL(n,\mathbb{R}) \cap O(n) .$$

2.1.9 The symplectic groups Sp(2n)

In GL(n,K), for $K = \mathbb{H}$, we write it as Sp(2n) and call it the *symplectic group*, $Sp(2n,\mathbb{H})$, the 4n(2n+1)-parameter group that leaves invariant the nondegenerate skew-symmetric bilinear form. Clearly, $Sp(2n,\mathbb{H}) \subset GL(n,\mathbb{H})$, and the matrices need not be unitary. Restriction to real matrices gives the n(2n+1)-parameter group $Sp(2n,\mathbb{R})$.

The symplectic group $Sp(2n) = U(4n) \cap Sp(2n,\mathbb{H})$ is known as the *unitary symplectic group* . This group looks like $Sp(2n,\mathbb{R})$, and is a n(2n+1)-parameter group. Note that symplectic groups exist only for even-dimensional vector spaces.

Section 2.2 Lie Algebras and Lie Groups

Lie groups are continuous groups, each of which forms a differentiable manifold with an analytic structure. We can establish a correspondence between the analytic subgroups of a Lie group and the subalgebras of its Lie algebra. ([11],[16],[21],[38],[39],[77]) Isomorphism of the Lie algebras is equivalent to *local* isomorphism of the corresponding Lie groups. The Lie algebra is defined by a commutator, and satisfies the anticommutative law and the Jacobi identity. (See the definitions below.)

2.2.1 Lie algebras

Some matrices in the general linear groups such as SO(n), SU(n), and Sp(2n) are not closed under ordinary matrix multiplication. For example, if

$$A = \begin{bmatrix} 0 & x \\ -x & 0 \end{bmatrix},$$ which is skew-symmetric,

then $$A^2 = \begin{bmatrix} -x^2 & 0 \\ 0 & -x^2 \end{bmatrix},$$ which is no longer skew-symmetric. Mathematicians have been trying to find some generalized " matrix products " to make these matrices closed under multiplication, which led naturally to the study of Lie , Jordan and other (nonassociative) algebras.

PROPOSITION : For $K \in \{\mathbb{R},\mathbb{C},\mathbb{H},\mathbb{O}\}$ and $A,B \in M_n(K)$, we define the bracket product $[A,B]$ as the commutator, i.e., $[A,B] = AB - BA$. Then SO(n), SU(n), and Sp(2n) are closed under $[\ ,\ ]$.

Thus, SO(n), SU(n) and Sp(2n) become algebras (over $\mathbb{R}$) with the bracket multiplication that has the following properties :

I $[A,B] = -[B,A]$ (anticommutative law) ,

II $[A,B+C] = [A,B] + [A,C]$,

$[A+B,C] = [A,C] + [B,C]$,

III for $r \in \mathbb{R}$, $r[A,B] = [rA,B] = [A,rB]$,

IV $[A,[B,C]] + [B,[C,A]] + [C,[A,B]] = 0$.

Property (IV) is called the *Jacobi identity*.

DEFINITION : A real vector space with a product satisfying I...IV is called a *Lie algebra*.

Let us consider low-dimensional Lie algebras. For dimension 1, the vector space is just $\mathbb{R}$, and making $X,Y \in \mathbb{R}$ we have

$$[X,Y] = X[1,Y] = XY[1,1] = 0 \qquad \text{(by I)}.$$

So we have the trivial product (which obviously satisfies I...IV). Consider $\mathbb{R}^2$ with basis e_1, e_2. We must have

$$[e_1,e_1] = [e_2,e_2] = 0 \quad \text{and} \quad [e_1,e_2] = -[e_2,e_1] .$$

Let $[e_1,e_2] = ae_1 + be_2$. Then, for example,

$$[e_1,[e_1,e_2]] = [e_1,(ae_1 + be_2)]$$
$$= a[e_1,e_1] + b[e_1,e_2]$$
$$= b(ae_1 + be_2) .$$

By the Jacobi identity,

$$[e_1,[e_1,e_2]] + [e_1,[e_2,e_1]] + [e_2,[e_1,e_1]]$$
$$= [e_1,[e_1,e_2]] - [e_1,[e_1,e_2]] + 0$$
$$= 0 .$$

So $b(ae_1 + be_2) + [e_1,(-ae_1 - be_2)] = 0$.

$b(ae_1 + be_2) - b(ae_1 + be_2) = 0$. Therefore, an identity.

The result is true with no conditions on a,b . If we take a=b=0 , we will get the trivial Lie algebra.

In the case $SO(3) = \{ \begin{bmatrix} 0 & a & b \\ -a & 0 & c \\ -b & -c & 0 \end{bmatrix} \mid a,b,c \in \mathbb{R} \}$

the Lie algebra has three dimensions, and the bases are e_1,e_2,e_3 , where

$$[e_1,e_2] = e_3 , \quad [e_2,e_3] = e_1 , \quad [e_3,e_1] = e_2 .$$

Clearly, the set e_1,e_2,e_3 has the 4 properties of a Lie algebra, viz.

I $[e_i,e_j] = -[e_j,e_i]$ for i = j and i,j = 1,2,3 ,
$[e_i,e_j] = 0$ for i = j and i,j = 1,2,3 ;

II $[e_1,(e_2+e_3)] = [e_1,e_2] + [e_1,e_3]$;

III for $r \in \mathbb{R}$, $r[e_1,e_2] = [re_1,e_2] = [e_1 + re_2]$;

IV $[e_1,[e_2,e_3]] + [e_2,[e_3,e_1]] + [e_3,[e_1,e_2]]$
$= [e_1,e_1] + [e_2,e_2] + [e_3,e_3]$
$= 0$.

2.2.2 Lie groups

Let G be a differentiable n-manifold which is also a group, and let the operations

$$G \times G \longrightarrow G \ , \qquad G \longrightarrow G$$

$$(a,b) \longmapsto ab \ , \qquad a \longmapsto a^{-1}$$

be smooth functions. Then G is called a *Lie group*.

Let G be a Lie group with identity element e, and suppose that X_e is a tangent vector at e ($X_e \in T_eG$). Then we can get a vector field defined on all of G as follows. For any $g \in G$ let $L_g : G \longrightarrow G$ be the diffeomorphism given by $L_g(X) = gX$ for each $X \in G$. This is called *left-translation* by g. We set $X_g = dL_g X_e$ (*) ($dL_g : T_eG \longrightarrow T_gG$) . Such vector fields are called *left-invariant* ; i.e., a vector field X on G is *left-invariant* if it satisfies (*) .

PROPOSITION : If X, Y are left-invariant vector fields on G, so is [X,Y].

Now one can see easily that X, Y be left-invariant implies that X+Y , and also that rX ($r \in \mathbb{R}$) are left invariant. Thus, the set of left-invariant fields on G becomes a subalgebra of the Lie algebra of all smooth vector fields. Since left-invariant vector fields correspond one-to-one with elements of T_eG, this Lie algebra is n-dimensional. We denote it by L(G) and call it the *Lie algebra* of G.

Section 2.3 Nonassociative Algebras and Nilpotent Algebras

The octonions, Lie algebras, and Jordan algebras are well-known cases of nonassociative algebras. (See, e.g., [38],[67].) They also possess special properties, some of which will be considered in this section.

2.3.1 Linear associative and nonassociative algebras

A *group* is a closed algebraic system with one internal composition law, satisfying some specified conditions. A *ring* is an algebraic system with two internal composition laws, one of which is distributive with respect to the other. A *vector space* has one internal plus one external composition law. We will now consider algebraic systems with two internal composition laws and one external composition law. Such a system combines the features of a ring and of a linear space, and is called a *linear algebra*.

DEFINITION : A *(linear) associative algebra* is an algebraic system $(V,F,\cdot)$, where V is a vector space, and F is a commutative field, that satisfies the following axioms:

I $\quad x + (y + z) = (x + y) + z$ for all $x,y,z \in V$.

II $\quad$ There exists an element $0 \in V$ such that $0 + x = x + 0 = x$ for all $x \in V$.

III $\quad$ For every $x \in V$ there exists an element $x \in V$ such that $x + (-x) = 0$.

IV $x + y = y + x$ for all $x, y \in W$.

V $x(yz) = (xy)z$ for all $x, y, z \in W$.

VI $x(y + z) = xy + xz$ and $(x + y)z = xz + yz$ for all $x, y, z \in W$.

VII $a(bx) = (ab)x$ for all $a, b \in \mathbb{R}$ and all $x \in W$.

VIII $1 \cdot x = x$ for all $x \in W$ (1 is the unit of F) .

IX $a(x + y) = ax + ay$ for all $a \in F$ and all $x, y \in W$.

X $(a + b)x = ax + by$ for all $a, b \in F$ and all $x \in W$.

XI $a(xy) = (ax)y = x(ay)$ of all $a \in F$ and all $x, y \in W$.

DEFINITION : A (linear) *nonassociative algebra* is an algebraic system which obeys all axioms of a linear (associative) algebra except that $x(yz) = (xy)z$ need not hold.

2.3.2 Jordan algebras

A *Jordan algebra* J is a nonassociative algebra with the Jordan product, $x \cdot y$, of elements that obeys the following two axioms :

I $x \cdot y = y \cdot x$ for all $x, y \in J$. (commutativity)

II $x^2 \cdot (y \cdot x) = (x^2 \cdot y) \cdot x$ for all $x, y \in J$. (Jordan identity)

Supposing A is a (linear) associative algebra, and we define $x \cdot y = \frac{1}{2}(xy+yx)$, then, this system is a Jordan algebra. ([49])

2.3.3 Alternative algebras

The *alternative algebra* A is defined by the identities

I Left alternative law : $x^2y = x(xy)$ for all $x,y \in A$.

II Right alternative law : $yx^2 = (yx)x$ for all $x,y \in A$.

Clearly, any associative algebra is alternative. In terms of the associator,

$$(x,y,z) = (xy)z - x(yz) ,$$

an alternative algebra A satisfies $(x,y,x) = 0$. The left and right alternative laws are, respectively, equivalent to

$$(x,x,y) = 0 \quad \text{and} \quad (y,x,x) = 0 \quad \text{for all} \quad x,y \in A.$$

In an alternative algebra, the associator (x_1,x_2,x_3) alternates in the sense that, for any permutation σ of 1,2,3 we have

$$(x_{1\sigma},X_{2\sigma},X_{3\sigma}) = (\text{sgn } \sigma)(x_1,x_2,x_3) \ ;$$

for example :

$$(x,y,z) = - (y,x,z) = (y,z,x)$$

for all $x,y,z \in A$. Also for alternative algebras, if any two of x,y,z are equal to each other, then $(x,y,z) = 0$. The relation

$$(xy)x = x(yx) \quad \text{for all} \quad x,y \in A$$

is called the *flexible law*. All Lie, Jordan, and alternative algebras are flexible.

2.3.4 Nilpotent algebras

The subalgebra generated by any two elements of an alternative algebra A is associative. An algebra A over a field F is called *power-associative* in case the subalgebra F(x) of A generated by any element x in A is associative. This is equivalent to defining power of a single element x in A recursively by

$$x^1 = x \quad , \quad x^{i+1} = xx^i \quad ,$$

and requiring that

$$x^i x^j = x^{i+j} \quad \text{for all } x \text{ in } A \quad (i,j = 1,2,3,\ldots).$$

An element x in a power-associative algebra A is called *nilpotent* in case there is an integer r such that $x^r = 0$. An algebra (ideal) consisting only of nilpotent elements is called a *nilalgebra* (*nilideal*). Any alternative nilalgebra A of finite dimension over a field F is nilpotent.

Section 2.4 Poisson Algebras, Heisenberg Groups and Hamiltonian Systems

The study of Poisson algebras of classical mechanics leads naturally to Lie algebras. Therefore, mathematical results from the theory of Lie algebras and Lie groups can be applied to dynamical problems that involve the Poisson bracket in classical mechanics. The Heisenberg group has been used to explain the relation of position and momentum in phase space. In a Hamiltonian system that depends on time and initial position (q,p), the motion is determined by the Hamiltonian function H(p,q). ([26],[29],[66],[75])

2.4.1 Poisson algebras

Let A be a (linear) associative algebra. An additional, nonassociative internal composition law, the *Poisson bracket*, $[x,y]_{PB}$, is introduced, that satisfies the following axioms (in addition to the already existing ones of x+y , ax, xy) :

i $[x,x]_{PB} = 0$, for all $x \in A$.

ii $[x,(y+z)]_{PB} = [x,y]_{PB} + [x,z]_{PB}$ and

$[[x+y],z]_{PB} = [x,z]_{PB} + [y,z]_{PB}$,

for all $x,y,z \in A$.

iii $a[x,y]_{PB} = [ax,y]_{PB} = [x,ay]_{PB}$, for all $x \in A$ and all $a \in \mathbb{R}$.

iv $[x,yz]_{PB} = [x,y]_{PB}z + y[x,z]_{PB}$,

for all $x,y,z \in A$.

Axioms (ii), (iii), and (iv) are mixed distributive and associative laws. We observe that axiom (i) is equivalent to $[x,y]_{PB} = -[y,x]_{PB}$ for all $x,y \in A$. In fact, the above axioms are the same as the previous axioms of a Lie algebra, if we define $[x,y]_{PB} = xy - yx$. Thus, the study of the structure of Poisson algebras leads quite naturally to the theory of Lie algebras. Poisson algebras first occurred in the Hamiltonian formulation of classical mechanics. Let A be the set of all infinitely differentiable real-valued function from $\mathbb{R} \times \mathbb{R} \longrightarrow \mathbb{R}$. (That is, A consists of real functions u(p,q) of two variables, and all partial derivatives, such as, $\frac{\partial u}{\partial p}$, $\frac{\partial u}{\partial q}$, $\frac{\partial^2 u}{\partial p^2}$, $\frac{\partial^2 u}{\partial q^2}$, $\frac{\partial^2 u}{\partial p \partial q}$, etc., exist.) Make this set into an ordinary associative algebra by pointwise sum, scalar product, and product:

$$(u + v)(p,q) = u(p,q) + v(p,q) ,$$

$$(au)(p,q) = au(p,q) ,$$

$$uv(p,q) = u(p,q)\, v(p,q) .$$

Now define the Poisson bracket :

$$[u,v]_{PB} = \frac{\partial u \partial v}{\partial q \partial p} - \frac{\partial u \partial v}{\partial p \partial q} .$$

Then, the Poisson algebra of classical mechanics has the structure of a Lie algebra.

2.4.2 The Heisenberg groups

The canonical commutation relations of quantum mechanics are given by

$$[q_j, q_k] = [p_j, p_k] = 0 \quad \text{and}$$

$$[p_j, q_k] = -[q_j, p_k] = -i\hbar\,\delta_{jk} \quad , \quad j,k = 1,\ldots,n \quad .$$

$$(p_j = -i\hbar\frac{\partial}{\partial x_j} \quad , \quad q_k = x_k \quad .)$$

Each element may be considered as a realization of the Lie algebra identities :

$$[P_j, Q_k] = -i\,\delta_{jk}E = -[Q_j, P_k] \quad ,$$

$$[P_j, E] = [Q_j, E] = 0 \quad . \qquad \text{-----} \; (2.4.1)$$

Note that $P_j = -i\hbar\frac{\partial}{\partial x_j}$, $Q_j = X_j$.

The formulation (2.4.1) refers to a basis P_j, Q_j , $j = 1,\ldots,n$, with

$$[P_j, P_k] = [Q_j, Q_k] = 0 \quad ,$$

$$[P_j, Q_k] = -[Q_k, P_j] = -\delta_{kj} \quad .$$

By using the Poisson bracket of classical mechanics, we obtain

$$[P_j, Q_k] = \sum \left[\frac{\partial P_j}{\partial Q_i}\frac{\partial Q_k}{\partial P_i} - \frac{\partial Q_k}{\partial Q_i}\frac{\partial P_j}{\partial P_i}\right]$$

$$= -\sum \frac{\partial Q_k}{\partial Q_i}\frac{\partial P_j}{\partial P_i}$$

$$= -\frac{\partial P_j}{\partial P_k}$$

$$= -\delta_{jk} \quad .$$

Since every P_j does not depend on all Q_k, we have

$$\frac{\partial P_j}{\partial Q_i}\frac{\partial Q_k}{\partial P_i} = 0 \qquad i,j,k = 1,\ldots,n \;.$$

Similarly :

$$[Q_k, P_j] = \delta_{jk} \quad \text{and} \quad [P_j, P_k] = [Q_j, Q_k] = 0 \;.$$

The groups formed by the commutators above are called the *Heisenberg groups*.

2.4.3 Hamiltonian systems

The position of a Hamiltonian system at time t is given by a pair of functions $(Q(t), P(t))$. The initial dynamical state of the system is given by $Q(0) = q$ and $P(0) = p$. In fact, the state of the system is determined by both the time t and the initial state (q,p), and so we write

$$Q = Q(t) = Q(t,p,q) \quad , \quad P = P(t) = P(t,p,q) \quad ,$$
$$Q(0,p,q) = q \quad , \quad P(0,p,q) = p \quad .$$

The motion of a Hamiltonian system is determined by a function $H(p,q)$, which is called the *Hamiltonian* of the system. The Hamiltonian is determined by the underlying physics. The dynamical behaviour of the system is determined by Hamilton's ordinary differential equations :

$$\frac{dQ}{dt} = \frac{\partial H(Q,P)}{\partial p} \quad , \quad \frac{dP}{dt} = -\frac{\partial H(Q,P)}{\partial q} \quad ,$$

where $$\frac{\partial H}{\partial q} = \left(\frac{\partial H}{\partial q_1}, \frac{\partial H}{\partial q_2}, \ldots, \frac{\partial H}{\partial q_n} \right),$$

$$\frac{\partial H}{\partial p} = \left(\frac{\partial H}{\partial p_1}, \frac{\partial H}{\partial p_2}, \ldots, \frac{\partial H}{\partial p_n} \right).$$

A simple example of a dynamical system is given by the harmonic oscillator. Let Q be the displacement of the mass from equilibrium and P be the momentum of the mass, and choose units so that all constants turn to be simple. This problem has one degree of freedom, so the variables are $(q_1,p_1) = (q,p)$. The Hamiltonian for this system can be chosen to be

$$H(q,p) = \frac{1}{2}(q^2 + p^2) .$$

Hamilton's canonical equations for this system are

$$\frac{dQ}{dt} = P \quad \text{and} \quad \frac{dP}{dt} = -Q . \qquad \text{----- (2.4.2)}$$

From (2.4.2) , $$\frac{d^2Q}{dt^2} = \frac{dP}{dt}$$

then $$\frac{d^2Q}{dt^2} + Q = 0 .$$

The solution of the initial value problem for Hamilton's equations is

$$Q(t) = q\cos(t) + p\sin(t) ,$$

$$P(t) = p\cos(t) - q\sin(t) .$$

Section 2.5 The Algebra and Calculus of Polarization

Light is a *transverse* electromagnetic wave. The vibrations of the electric and magnetic fields take place in a plane perpendicular to the direction of propagation of the wave. The properties of the electromagnetic wave depend on the orientation of the electric and magnetic vectors **E** and **B**, which is associated with the *polarization*. Electromagnetic waves with any polarization can be represented as a superposition of two linearly polarized wave whose electric vectors oscillate in mutually orthogonal planes. Hence it can be stated that electromagnetic waves have two independent states of polarization. ([14],[33],[34])

2.5.1 Linear polarization

If in the process of wave propagation the vector **E** remains in the same plane parallel to the direction of propagation, the waves are called *linearly polarized*.

Let us consider the superposition of two linearly polarized waves having the same frequency and propagation in the same direction. We shall assume that the oscillations of **E** of the first wave lie in the XZ-plane and of the second wave in the YZ-plane (Fig. 2.5A). Then we can write

$$\mathbf{E}_x(z,t) = \mathbf{i}\ E_{ox}\cos(kz-\omega t)\ ,$$

and $$\mathbf{E}_y(z,t) = \mathbf{j}\ E_{oy}\cos(kz-\omega t+\varepsilon)\ ,$$

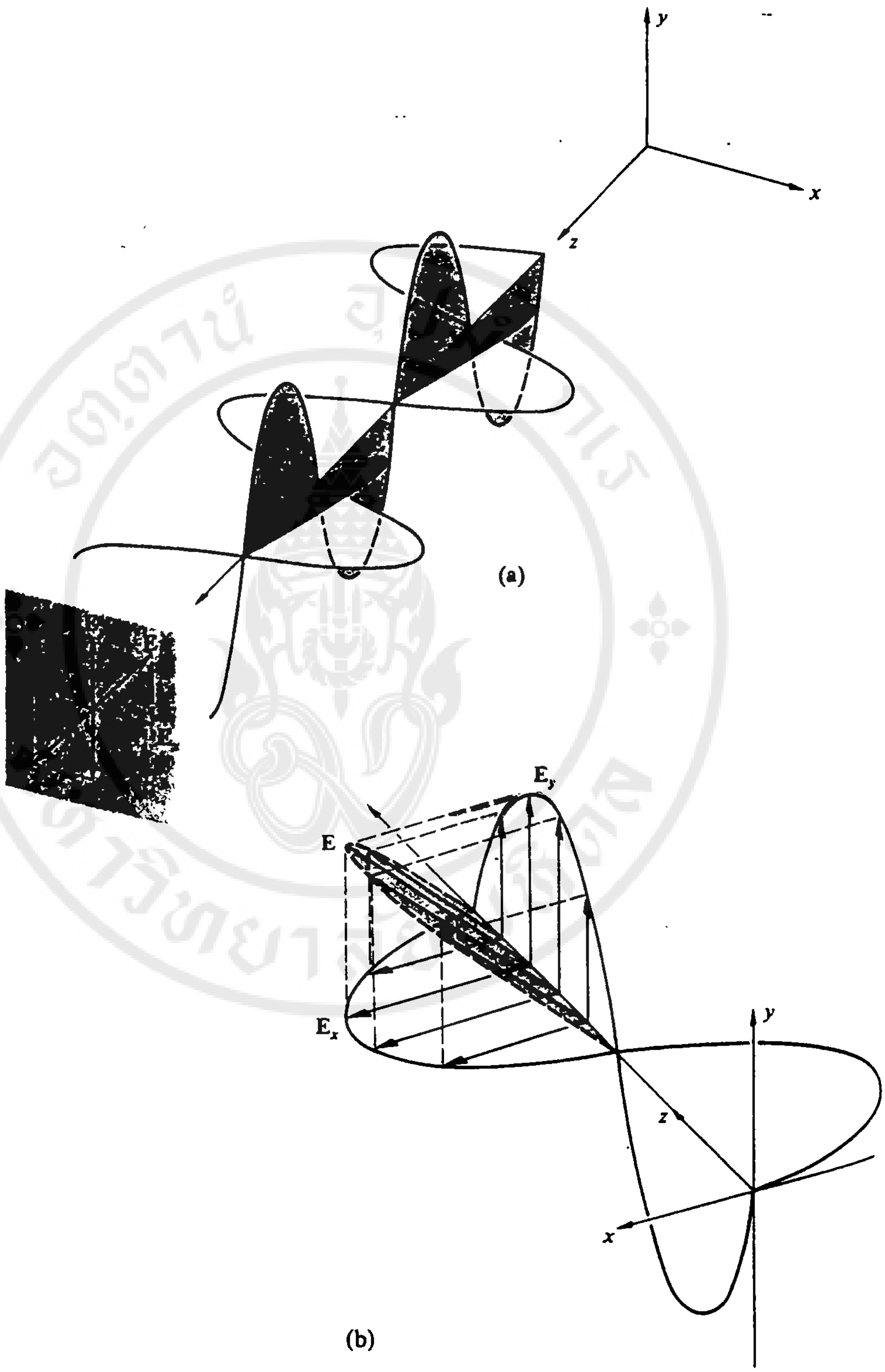
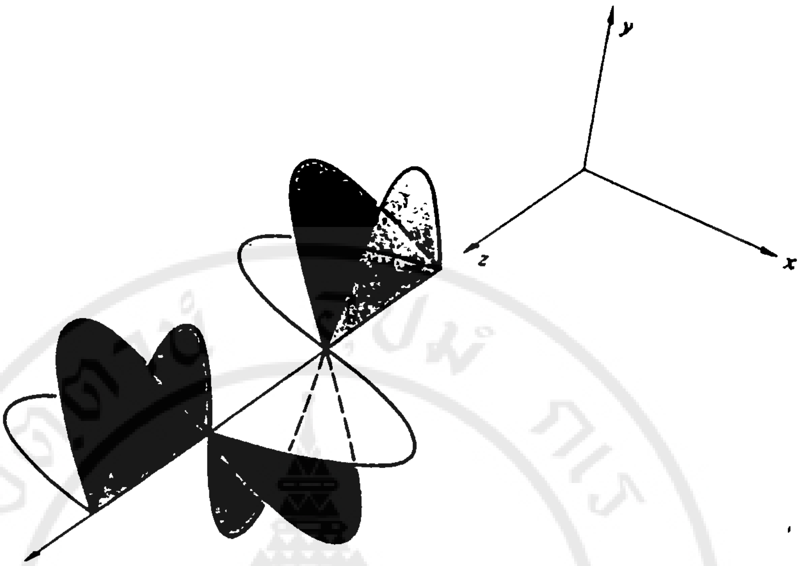


Fig. 2.5A Linearly polarized light



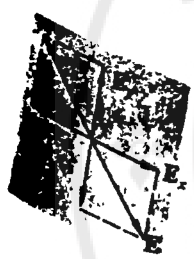


Fig. 2.5B Vibration of E in the plane of polarization

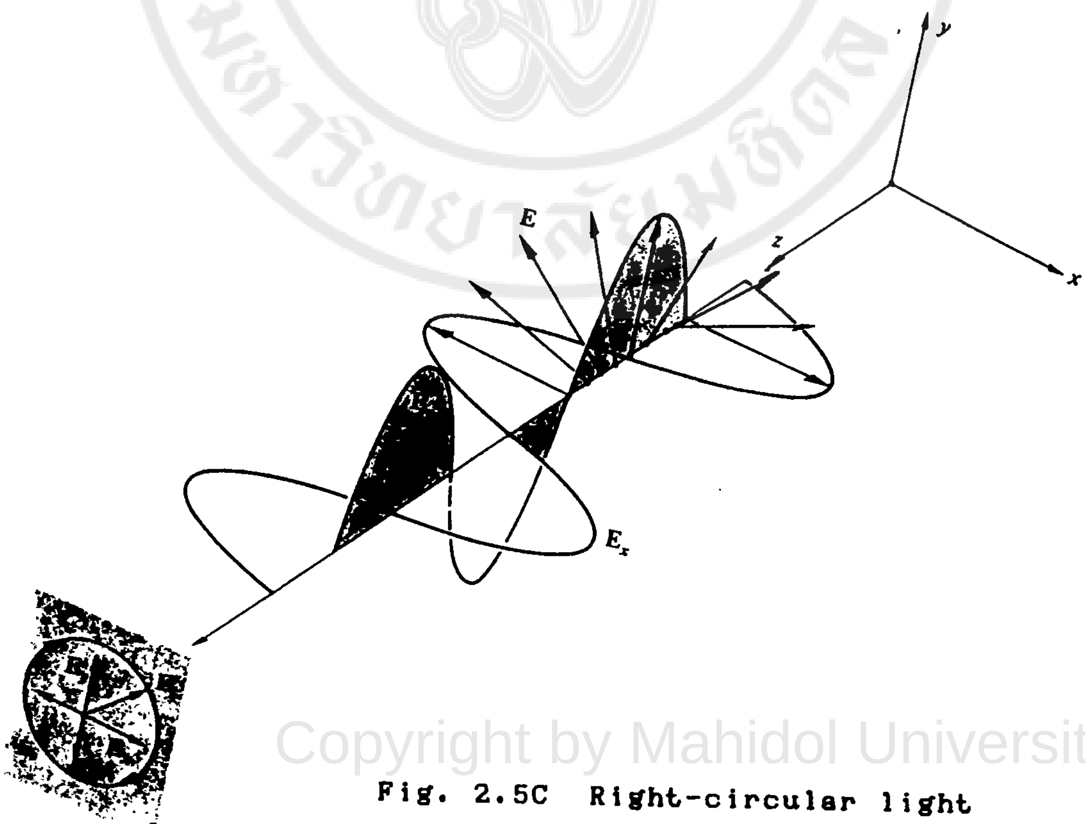


Fig. 2.5C Right-circular light

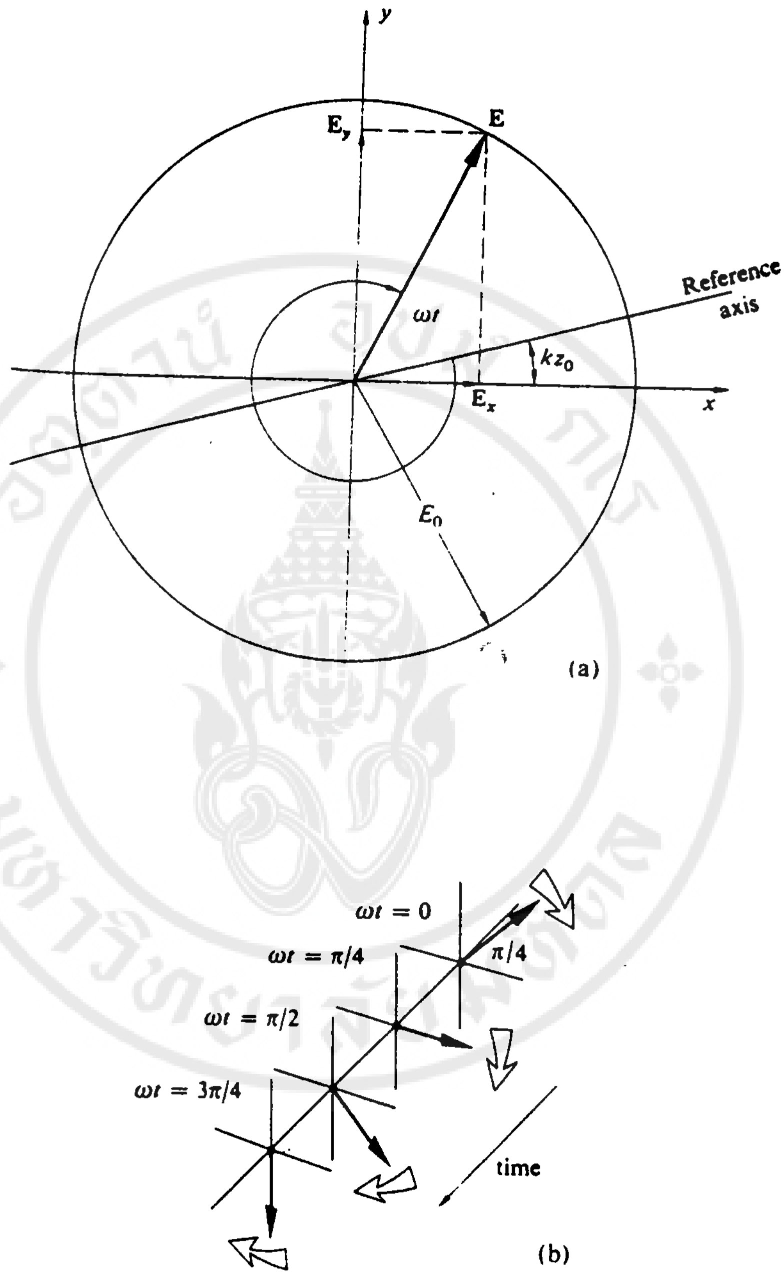


Fig. 2.5D Rotation of the electric vector in a right-circular wave (Note that the rotation rate is ω and kz = π/4 .)

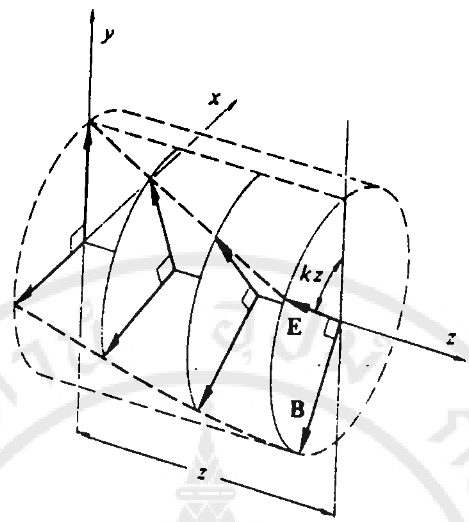


Fig. 2.5E Rotation of E in right-circular light

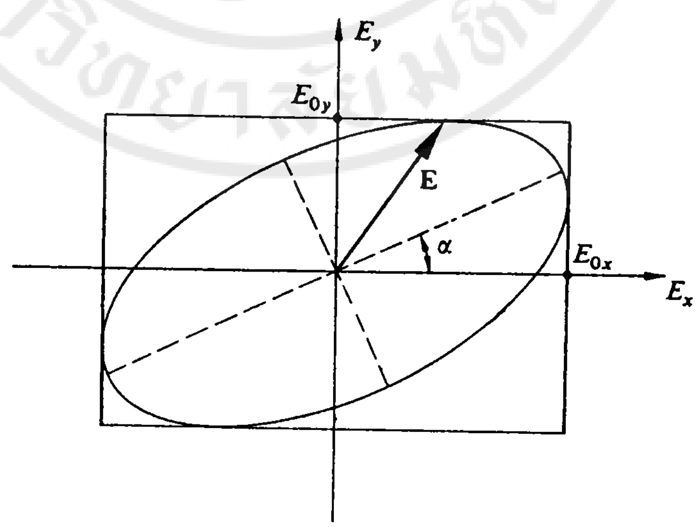


Fig. 2.5F Elliptically polarized light

where ε is the relative phase difference between the wave. Let us analyze the resultant optical disturbance

$$\mathbf{E}(z,t) = \mathbf{E}_x(z,t) + \mathbf{E}_y(z,t) \quad \text{----- (2.5.1)}$$

If ε is zero or $\pm(2m)\pi$, where $m \in \mathbf{N}$, the waves are said to be in phase. In that particular case Eq.(2.5.1) becomes

$$\mathbf{E}(z,t) = (\mathbf{i}\, E_{ox} + \mathbf{j}\, E_{oy}) \cos(kz-\omega t) .$$

Then the new amplitude of the resultant wave is $(\mathbf{i}\, E_{ox} + \mathbf{j}\, E_{oy})$. It is linearly polarized, as shown in Fig. 2.5A . Therefore we can describe any plane-polarized light with the plane of vibration rotated (and not necessarily by 90°) from that of the previous condition, as indicated in Fig. 2.5B .

2.5.2 Circular polarization

When both constituent waves have equal amplitudes, i.e. $E_{ox} = E_{oy} = E_o$, and in addition, their relative phase difference is $\varepsilon = -(\pi/2) + (2m)\pi$, where m is an integer, we have

$$\mathbf{E}_x(z,t) = \mathbf{i}\, E_o \cos(kz-\omega t)$$

and $$\mathbf{E}_y(z,t) = \mathbf{j}\, E_o \sin(kz-\omega t) .$$

The consequent wave is given by

$$\mathbf{E} = E_o(\mathbf{i}\cos(kz-\omega t) + \mathbf{j}\sin(kz-\omega t)) \quad \text{----- (2.5.2)}$$

(Fig. 2.5C). Now the scalar amplitude is a constant E_o . But the direction of $\mathbf{E}$ is time varying and it is not restricted as before to a single plane. Fig. 2.5D depicts what is happening at some arbitrary point z_o on the axis. At $t = 0$ $\mathbf{E}$ lie along the reference axis in Fig. 2.5D(a) and so

$$\mathbf{E}_x = \mathbf{i}\, E_o \cos(kz_o) \quad \text{and} \quad \mathbf{E}_y = \mathbf{j}\, E_o \sin(kz_o) \,.$$

At a later time, $t = (kz_o/\omega)$, $\mathbf{E}_x = \mathbf{i}E_o$, $\mathbf{E}_y = 0$, and

$\mathbf{E}$ is along the x axis. The resultant electric field vector $\mathbf{E}$ is rotating clockwise at an angular frequency ω. Such a wave is said to be *right-circularly polarized* (Fig. 2.5E), and one simply refers to it as *right-circular* light. If $\varepsilon = (\pi/2) + (2m)\pi$ where m is integer then

$$\mathbf{E} = E_o\,(\,\mathbf{i}\cos(kz-\omega t) - \mathbf{j}\sin(kz-\omega t)\,) \,. \quad \text{----(2.5.3)}$$

The amplitude is the same as the previous amplitude but $\mathbf{E}$ now rotates counter-clockwise, and the wave is referred to as *left-circularly polarized*, or, more simply, as *left-circular* light.

A *linearly polarized* wave can be synthesized from two opposite circularly polarized waves of equal amplitudes. In particular, if we add the right-circular wave of Eq.(2.5.2) to the left-circular wave of Eq.(2.5.3) we get

$$\mathbf{E} = \mathbf{i}\ 2E_o\cos(kz-\omega t)$$

which has a constant amplitude vector of $\mathbf{i}(2E_o)$ and is therefore linearly polarized.

2.5.3 Elliptical polarization

In general, the resultant electric field vector **E** will both rotate and change its magnitude as well. We can write an expression for the curve traversed by the tip of **E**

$$E_x = E_{ox}\cos(kz-\omega t) \qquad \text{----- (2.5.4)}$$

and

$$E_y = E_{oy}\cos(kz-\omega t+\varepsilon)\ .$$

The equation of the curve we are looking for should be a function of neither position nor time, i.e. we should be able to get rid of the (kz-ωt) dependence. Expand the expression for E_y into

$$\frac{E_y}{E_{oy}} = \cos(kz-\omega t)\cos(\varepsilon) - \sin(kz-\omega t)\sin(\varepsilon)\ ,$$

and combine it with $\frac{E_x}{E_{ox}}$ to yield

$$\frac{E_y}{E_{oy}} - \frac{E_x}{E_{ox}}\cos(\varepsilon) = -\sin(kz-\omega t)\sin(\varepsilon)\ . \qquad \text{----- (2.5.5)}$$

From Eq.(2.5.4) $$\frac{E_x}{E_{ox}} = \cos(kz-\omega t)$$

then $$\sin(kz-\omega t) = \sqrt{1 - \frac{E_x^2}{E_{ox}^2}} \quad .$$

Substituting $\sin(kz - \omega t)$ into Eq.(2.5.5) and squaring both sides, we get

$$\left[\frac{E_y}{E_{oy}} - \frac{E_x}{E_{ox}}\cos(\varepsilon)\right]^2 = \left[1 - \frac{E_x^2}{E_{ox}^2}\right]\sin^2(\varepsilon) \quad .$$

Finally, on rearranging terms we have

$$\frac{E_y^2}{E_{oy}^2} + \frac{E_x^2}{E_{ox}^2} - \frac{2E_y E_x\cos(\varepsilon)}{E_{oy}E_{ox}} = \sin^2(\varepsilon) \quad . \text{ --- (2.5.6)}$$

This is the equation of an ellipse making an angle α with the (E_x,E_y) coordinate system (Fig. 2.5F) such that

$$\tan(2\alpha) = \frac{2E_{ox}E_{oy}\cos(\varepsilon)}{E_{ox}^2 - E_{oy}^2} \quad .$$

If $\varepsilon = (2m-1)(\pi/2)$ and $m \in \mathbb{N}$, then Eq.(2.5.6) becomes

$$\frac{E_x^2}{E_{ox}^2} + \frac{E_y^2}{E_{oy}^2} = 1 \quad .$$

Furthermore, if $E_{oy} = E_{ox} = E_o$, this is reduced to

$$E_y^2 + E_x^2 = E_o^2 \quad ,$$

which, in agreement with our previous result, is a circle. If ε is an even multiple of π, the result of Eq.(2.5.6) is

$$\frac{E_x^2}{E_{ox}^2} - \frac{E_y^2}{E_{oy}^2} = 0 \ .$$

Therefore $$E_y = \frac{E_{oy}}{E_{ox}} E_x \ ;$$

and, similarly, for odd multiples of π,

$$E_y = - \frac{E_{oy}}{E_{ox}} E_x \ .$$

These are both straight lines having slope of $\pm \frac{E_{oy}}{E_{ox}}$, i.e. we have *linear light*. From the above, notice that we can make Eq.(2.5.6) into circular equation and linear equation, so that we may consider linear and circular light to be special cases of elliptically polarized light and give its specific state of plane polarized light in a P-state, while right- or left-circular light is in an R- or L-state, respectively.

2.5.4 The Stokes parameters

We consider a pair of plane waves which are orthogonal to each other and not necessarily monochromatic to be represented by the equations

$$E_x(t) = i\, E_{ox}(t) \cos(kz-\omega t+\varepsilon_x(t))$$
$$E_y(t) = j\, E_{oy}(t) \cos(kz-\omega t+\varepsilon_y(t)) \quad , \qquad \text{-----}\ (2.5.7)$$

where $E_{ox}(t)$ and $E_{oy}(t)$ are instantaneous amplitudes, ω is the instantaneous angular frequency, and $\varepsilon_x(t)$ and $\varepsilon_y(t)$ are the instantaneous phase factors.

We consider Eq.(2.5.6), which is valid only at a given instant of time

$$\frac{E_x(t)^2}{E_{ox}(t)^2} + \frac{E_y(t)^2}{E_{oy}(t)^2} - \frac{2E_x(t)E_y(t)\cos(\varepsilon(t))}{E_{ox}(t)E_{oy}(t)} = \sin^2(\varepsilon(t)) \quad , \qquad \text{-----}\ (2.5.8)$$

where $\varepsilon(t) = \varepsilon_x(t) + \varepsilon_y(t)$.

If we have *monochromatic* radiation, the amplitudes and phase are constant for all time, so Eq.(2.5.8) is reduced to

$$\frac{E_x(t)^2}{E_{ox}(t)^2} + \frac{E_y(t)^2}{E_{oy}(t)^2} - \frac{2E_x(t)E_y(t)\cos(\varepsilon)}{E_{ox}(t)E_{oy}(t)} = \sin^2(\varepsilon) \quad . \qquad \text{-----}\ (2.5.9)$$

In order to represent Eq.(2.5.7) in terms of the observables of the electromagnetic field, we must take a time average of Eq.(2.5.9)

$$\frac{\langle E_x(t)^2\rangle}{E_{ox}(t)^2} + \frac{\langle E_y(t)^2\rangle}{E_{oy}(t)^2} - \frac{2\langle E_x(t)\rangle\langle E_y(t)\rangle\cos(\varepsilon)}{E_{ox}(t)E_{oy}(t)} = \sin^2(\varepsilon) \quad , \qquad \text{----- (2.5.10)}$$

where $\langle E_i(t)E_j(t)\rangle = T^{-1}{}_0\int^T E_i(t)E_j(t)\,dt$, $i,j = x,y$. From Eq.(2.5.7), we can find average values indicated in Eq.(2.5.10)

$$\langle E_x^2(t)\rangle = \frac{1}{2} E_{ox}^2 \ ,$$

$$\langle E_y^2(t)\rangle = \frac{1}{2} E_{oy}^2 \ , \qquad \text{----- (2.5.11)}$$

$$\langle E_x(t)\rangle\langle E_y(t) = \frac{1}{2} E_{ox}E_{oy}\cos(\varepsilon) \quad .$$

Substituting Eq.(2.5.11) into Eq.(2.5.10) , and multiplying by $4E_{ox}^2E_{oy}^2$, we get

$$(2E_{ox}^2E_{oy}^2) + (2E_{ox}^2E_{oy}^2) - (2E_{ox}E_{oy}\cos(\varepsilon))^2$$
$$= (2E_{ox}E_{oy}\sin(\varepsilon))^2 \ . \qquad \text{------ (2.5.13)}$$

We now define the *Stokes parameters* in terms of four quantities inside the parentheses as

$$S_0 = \langle E_{ox}^2\rangle + \langle E_{oy}^2\rangle \ ,$$
$$S_1 = \langle E_{ox}^2\rangle - \langle E_{oy}^2\rangle \ ,$$
$$S_2 = \langle 2E_{ox}E_{oy}\cos(\varepsilon)\rangle \ ,$$
$$S_3 = \langle 2E_{ox}E_{oy}\sin(\varepsilon)\rangle \ ,$$

and write Eq.(2.5.13) as $S_0^2 = S_1^2 + S_2^2 + S_3^2$. We can explain the meaning of these parameters that

Table 2.5G Stokes and Jones vectors for some polarization states

State of Polarization	Stokes vectors	Jones vectors
Horizontal $\mathscr{P}$-state	$\begin{bmatrix} 1 \\ 1 \\ 0 \\ 0 \end{bmatrix}$	$\begin{bmatrix} 1 \\ 0 \end{bmatrix}$
Vertical $\mathscr{P}$-state	$\begin{bmatrix} 1 \\ -1 \\ 0 \\ 0 \end{bmatrix}$	$\begin{bmatrix} 0 \\ 1 \end{bmatrix}$
$\mathscr{P}$-state at $+45°$	$\begin{bmatrix} 1 \\ 0 \\ 1 \\ 0 \end{bmatrix}$	$\frac{1}{\sqrt{2}} \begin{bmatrix} 1 \\ 1 \end{bmatrix}$
$\mathscr{P}$-state at $-45°$	$\begin{bmatrix} 1 \\ 0 \\ -1 \\ 0 \end{bmatrix}$	$\frac{1}{\sqrt{2}} \begin{bmatrix} 1 \\ -1 \end{bmatrix}$
$\mathscr{R}$-state	$\begin{bmatrix} 1 \\ 0 \\ 0 \\ 1 \end{bmatrix}$	$\frac{1}{\sqrt{2}} \begin{bmatrix} 1 \\ -i \end{bmatrix}$
$\mathscr{L}$-state	$\begin{bmatrix} 1 \\ 0 \\ 0 \\ -1 \end{bmatrix}$	$\frac{1}{\sqrt{2}} \begin{bmatrix} 1 \\ i \end{bmatrix}$

S_0 is the total intensity of the radiation

S_1 is a reflection of a tendency for the polarization to resemble more to either a horizontal P-state (whereupon $S_1 > 0$) or a vertical one (in which case $E_1 < 0$) or there is no preferential orientation with respect to these axes ($S_1 = 0$). It may be elliptical at $\pm 45^{\circ}$, circular, or unpolarized.

S_2 is a P-state oriented either in the direction of $+45^{\circ}$ (when $S_2 > 0$) or in the direction of -45° when ($S_2 < 0$) or neither ($S_2 = 0$).

S_3 is the beam toward right-handedness ($S_3 > 0$), left-handedness ($S_3 < 0$) or neither ($S_3 = 0$) .

We show this in Table 2.5G .

2.5.5 The Jones vectors

The Jones vectors were invented by R. Clark Jones in 1941. Their role complements that of the Stokes parameters, and is only applicable to polarized waves. In that case it would seem that the most natural way to represent the beam would be in terms of the electric vector itself. Written in columm form, the Jones vector is

$$\mathbf{E} = \begin{bmatrix} E_x(t) \\ E_y(t) \end{bmatrix} , \qquad ----- (2.5.14)$$

where $E_x(t)$ and $E_y(t)$ are instantaneous scalar components of $\mathbf{E}$. If we preserve the phase information, we will be able to handle coherent waves. With this in mind, we rewrite Eq.(2.5.14) as

$$\mathbf{E} = \begin{bmatrix} E_{ox}(t)e^{i\phi_x} \\ E_{oy}(t)e^{i\phi_y} \end{bmatrix} , \qquad \text{----- (2.5.15)}$$

where ϕ_x and ϕ_y are the appropriate phase (ϕ=kz-wt).

We will consider the Jones vectors in 3 cases.

Case I Linear polarized light

The horizontal and vertical P-states are given by

$$\mathbf{E}_h = \begin{bmatrix} E_{ox}e^{i\phi_x} \\ 0 \end{bmatrix}$$

and

$$\mathbf{E}_v = \begin{bmatrix} 0 \\ E_{oy}e^{i\phi_y} \end{bmatrix} .$$

Therefore

$$\mathbf{E} = \begin{bmatrix} E_{ox}e^{i\phi_x} \\ E_{oy}e^{i\phi_y} \end{bmatrix} . \qquad \text{----- (2.5.15)}$$

If $E_{ox} = E_{oy}$ and $\phi_x = \phi_y$, then

$$\mathbf{E} = E_{ox}e^{i\phi_x} \begin{bmatrix} 1 \\ 1 \end{bmatrix} ,$$

which is a P-state at + 45°. In this case, the phase angle ϕ_x is zero and we can normalize the irradiance to unity. This is done by dividing both elements in the vector by the same scalar (real or complex) quantity so that the sum of the squares of the components is one. For example,

$$E_{+45} = \frac{1}{\sqrt{2}}\begin{bmatrix} 1 \\ 1 \end{bmatrix} \quad \text{and} \quad E_{-45} = \frac{1}{\sqrt{2}}\begin{bmatrix} 1 \\ -1 \end{bmatrix} ;$$

similarly, in normalized form,

$$E_h = \begin{bmatrix} 1 \\ 0 \end{bmatrix} \quad \text{and} \quad E_v = \begin{bmatrix} 0 \\ 1 \end{bmatrix} .$$

Case II Circular polarization

For this state of polarization $E_{ox} = E_{oy}$. If the y-component is leading the x-component by 90°, then it is called *right-circular* light

$$E_R = \begin{bmatrix} E_{ox}e^{i\phi_x} \\ E_{ox}e^{i(\phi_x - \pi/2)} \end{bmatrix} = E_{ox}e^{i\phi_x}\begin{bmatrix} 1 \\ -i \end{bmatrix} ;$$

hence the normalized Jones vector is

$$E_R = \frac{1}{\sqrt{2}}\begin{bmatrix} 1 \\ -i \end{bmatrix} ;$$ and, similarly, for

the left-circular light $$E_L = \frac{1}{\sqrt{2}}\begin{bmatrix} 1 \\ i \end{bmatrix} .$$

The sum of the components E_R and E_L gives the horizontal P-state

$$E = \frac{1}{\sqrt{2}}\begin{bmatrix} 2 \\ 0 \end{bmatrix} = \sqrt{2}\begin{bmatrix} 1 \\ 0 \end{bmatrix} .$$

Case III Elliptic polarization

We can put the circular form of polarization into an *elliptic* one by multiplying either component by a scalar. Thus

$$E_{ellipse} = \begin{bmatrix} s \\ -i \end{bmatrix} \text{ or } \begin{bmatrix} 1 \\ -is \end{bmatrix}, \text{ where } s \text{ is a scalar,}$$

describes one possible form of horizontal, right-handed, *elliptical* light.

Two vectors A and B are said to be orthogonal when $A \cdot B = 0$. Similarly two complex vectors are orthogonal when $A \cdot B^* = 0$. One refer to polarization states as being orthogonal when their Jones vectors are orthogonal. For example

$$E_R \cdot E_L^* = \frac{1}{\sqrt{2}}\begin{bmatrix} 1 \\ -i \end{bmatrix} \cdot \frac{1}{\sqrt{2}}\begin{bmatrix} 1 \\ i \end{bmatrix}^*$$

$$= \frac{1}{2}(1 + i^2) = 0 ,$$

or $$E_h \cdot E_v^* = \begin{bmatrix} 1 \\ 0 \end{bmatrix} \cdot \begin{bmatrix} 0 \\ 1 \end{bmatrix}^* = 0 .$$

Notice that $\quad E_R \cdot E_R^* = E_L \cdot E_L^* = 1 \quad ,$

and $\quad E_R \cdot E_L^* = E_L \cdot E_R^* = 0 \quad .$

This property is called *orthonormality*.

2.5.6 The Jones and Mueller matrices

Suppose that we have a polarized incident beam represented by its Jones vector E_i which passes through an optical element, emerging as a new vector E_t corresponding to the transmitted wave. The optical element has transformed E_i into E_t, by $E_t = AE_i$ where A is the *transformation matrix* of the optical element :

$$A = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} .$$

Suppose that E_i represents a P-state at $+ 45^\circ$ which passes through a quarter-wave plate whose fast axis is vertical (i.e. in the y-direction). The polarization state of the emergent wave is found as follows. We drop the constant-amplitude factors for convenience, and thus obtain,

$$AE_i = \begin{bmatrix} 1 & 0 \\ 0 & -i \end{bmatrix} \begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 \\ -i \end{bmatrix} = E_t \quad ,$$

the beam being right-circular, and A being the quarter-wave plate with the fast axis vertical. If the wave passes through a series of optical elements represented by the matrices $\quad A_1, A_2, A_3, \ldots, A_n \quad ,$ then

$$E_t = A_n \ldots A_3 A_2 A_1 E_i \quad .$$

Table 2.5H Jones and Mueller matrices

Linear optical element	Jones matrix	Mueller matrix
Horizontal linear polarizer ↔	$\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$	$\frac{1}{2}\begin{bmatrix} 1 & 1 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$
Vertical linear polarizer ↕	$\begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$	$\frac{1}{2}\begin{bmatrix} 1 & -1 & 0 & 0 \\ -1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$
Linear polarizer at +45° ⤢	$\frac{1}{2}\begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}$	$\frac{1}{2}\begin{bmatrix} 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$
Linear polarizer at −45° ⤡	$\frac{1}{2}\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}$	$\frac{1}{2}\begin{bmatrix} 1 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 \\ -1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$
Quarter-wave plate fast axis vertical	$e^{i\pi/4}\begin{bmatrix} 1 & 0 \\ 0 & -i \end{bmatrix}$	$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 \end{bmatrix}$
Quarter-wave plate fast axis horizontal	$e^{i\pi/4}\begin{bmatrix} 1 & 0 \\ 0 & i \end{bmatrix}$	$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 \end{bmatrix}$
Homogeneous circular polarizer right ↻	$\frac{1}{2}\begin{bmatrix} 1 & i \\ -i & 1 \end{bmatrix}$	$\frac{1}{2}\begin{bmatrix} 1 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 1 \end{bmatrix}$
Homogeneous circular polarizer left ↺	$\frac{1}{2}\begin{bmatrix} 1 & -i \\ i & 1 \end{bmatrix}$	$\frac{1}{2}\begin{bmatrix} 1 & 0 & 0 & -1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ -1 & 0 & 0 & 1 \end{bmatrix}$

In general, the matrices do not commute, and thus they must be applied in the proper order. It should be noted that the Jones calculus is of use only for computing results with light that is initially polarized in some way. There is no Jones vector representation for unpolarized light.

In 1943 Hans Mueller introduced a 4 x 4 matrix for dealing with the Stokes vectors. The Mueller matrices are applied in many ways rather similar to the Jones matrices. The basic cases of the Jones and Mueller matrices for the description of linear optical elements are given in Table 2.5H. Later on, physicists have found that the Mueller matrices can easily deal with coherent waves while the Jones matrices can not. For further details on the Stokes and Jones vectors for the polarization states and on the Jones and Mueller matrices for linear optical elements, see [33] and Tables 2.5G and 2.5H. A typical relationship between the Stokes vectors and the Mueller matrices is as follows :

$$S_t = \mathbf{M}_n \ldots \mathbf{M}_2 \mathbf{M}_1 S_i \quad ,$$

where

S_t stands for the Stokes vector of the transmitted wave,

S_i for that of the incident wave, and

$\mathbf{M}_k$ is the Mueller matrix for the k^{th} linear optical element, $k = 1,2,\ldots,n$.

Section 2.6 Spinors and Twistors

In this section, we survey the representations of the spin states of a relativistic particle in space, and in space-time, by introducing stereographic projections, the Riemann sphere, and holomorphic functions of spinors and twistors. Like the more familiar vectors and tensors, spinors and twistors are now playing very important roles in the mathematical analysis of the space-time geometries of relativity and relativistic quantum mechanics. ([9],[10],[12],[35],[60],[62],[63])

2.6.1 Spinors

Spinors are extremely important in the study of spin states of a particle in relativistic quantum mechanics. The algebraic and geometric orders of the complex number system is clearly reflected in the quantum superposition law. The two-dimensionality of the complex number system is seen to be intimately related to the three-dimensionality of physical space. The standard form of complex numbers is $\zeta = x + iy$, (x,y real), where i is the imaginary unit ($i = \sqrt{-1}$), with its axis perpendicular to the real axis. The complex number ζ is located in a plane rather than on a line. It is natural to associate the complex number ζ with a point(x,y) in the plane whose rectangular coordinates are x and y. Each complex number corresponds to just one point, and conversely.

The geometry of the complex number ζ is displayed in Fig. 2.6A. The xy plane is called the Argand plane, or ζ-plane.

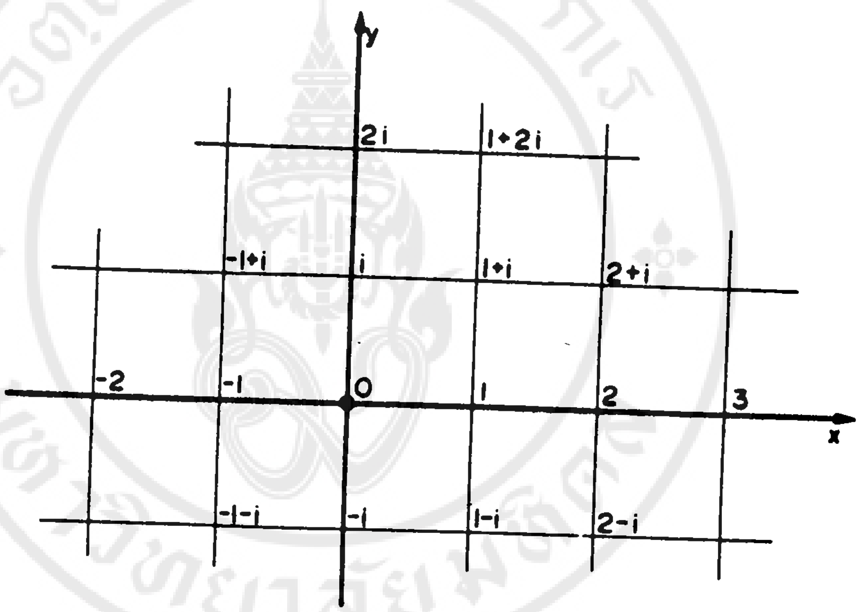


Fig. 2.6A The representation of complex numbers on the Argand plane

The complex number ζ can also be represented by a vector that lies in the ζ-plane. The product of two complex numbers, neither a scalar nor a vector, is used in place of ordinary vector analysis.

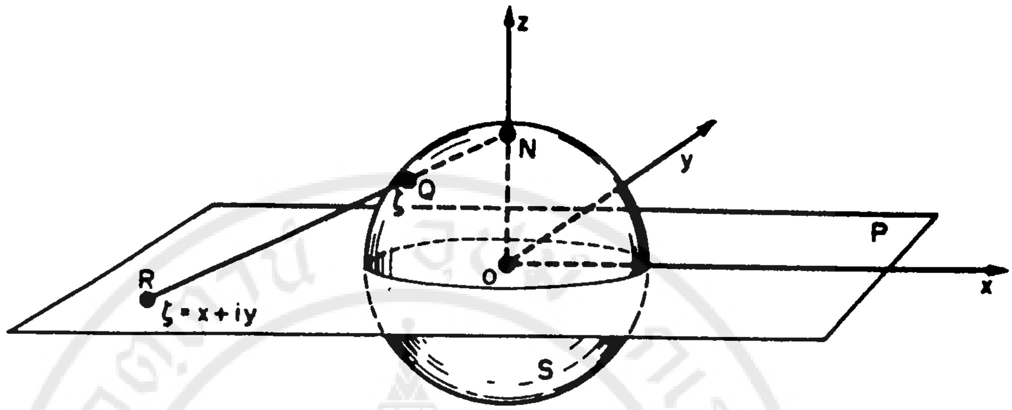


Fig. 2.6B Stereographic projection from the Riemann sphere to the Argand plane

We next consider how complex numbers can be used to represent the spin state of an electron in space-time with three dimensions of space and one of time. One basic rule of quantum mechanics is the *linear superposition* of quantum states. Let A and B be the states of spin of an electron with the direction of the spin vector pointing upward and downward, respectively. Then any other spin direction can be represented by the superposition $\alpha A + \beta B$, where α and β are complex numbers (not both zero), and we get a physically distinct situation for each distinct ratio $\alpha:\beta$. There is thus a one-to-one correspondence between the direction in space and the ratio $\alpha:\beta$ of a pair of complex numbers. It is convenient to include with the complex plane the point at infinity and to use $\beta = 0$. The complex plane together with this point at infinity is called the *extended* complex plane. Construct a third axis (the z-axis) perpendicular to the plane P and through the centre O (see Fig. 2.6B). The complex plane P passes through the equator of a unit sphere centred at the point $\zeta = 0$. The north pole N of S is the place where the positive z-axis meets S ; the point N of the sphere S corresponds to the point at infinity. Consider a point R of P, representing the complex number ζ. The intersection of the line from the point R to the north pole N and the surface of sphere S is a unique point Q. This point Q then represents the complex numbers ζ, on S, where ζ is now viewed as a ratio $\alpha:\beta$. The one-to-one correspondence between the points of the sphere and the points of the extended complex plane is called the *stereographic projection*, and the sphere is known as the *Riemann sphere*. The Riemann sphere S represents the space of the ratio $\alpha:\beta$. The importance of the one-to-one correspondence for our purpose is that it enables us to easily imagine the four-dimensional situation of the states of spin of an electron.

The geometric correspondence between ζ and ζ' can be explained in more physical terms as the relativistic correspondence between two observers. One observes a star and assigns a complex number ζ to each star for its position. The other observer, in any state of motion relative to the first, obtains a different complex number ζ' to label the same star. The correspondence between ζ and ζ' is then given by a bilinear holomorphic transformation

$$\zeta' = \frac{A_{11}\zeta + A_{12}}{A_{21}\zeta + A_{22}} , \qquad \text{----- (2.6.1)}$$

where A_{11}, A_{12}, A_{21}, and A_{22} are complex parameters defining the relative motion and orientations of the two observers. The formula (2.6.1) is analytically expressed entirely in terms of $\zeta = x + iy$ or its complex conjugate $\bar{\zeta} = x - iy$. Eq.(2.6.1) can also be rewritten formally as

$$\zeta' = \mathbf{A}\cdot\zeta := (A_{11}\zeta + A_{12})(A_{21}\zeta + A_{22})^{-1} , \qquad \text{----- (2.6.2)}$$

where $\mathbf{A} = \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix}$ is a non-singular matrix,

and thus $\det \mathbf{A} \neq 0$. From the property of an inverse $\mathbf{A}^{-1}$ of a matrix $\mathbf{A}$:

$$\mathbf{A}\mathbf{A}^{-1} = \mathbf{I} , \qquad \text{----- (2.6.3)}$$

we have

$$\mathbf{A}^{-1} = \lambda \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix} ,$$

where $\lambda = \dfrac{1}{A_{11}A_{22} - A_{21}A_{12}}$.

On substituting the matrices $\mathbf{A}$ and $\mathbf{A}^{-1}$ into Eq.(2.6.3), we obtain

$$\mathbf{A}\mathbf{A}^{-1} = \lambda \begin{bmatrix} A_{22} & -A_{12} \\ -A_{21} & A_{11} \end{bmatrix} \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix} ,$$

$$I := \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = \lambda \begin{bmatrix} A_{22}A_{11} - A_{12}A_{21} & A_{22}A_{12} - A_{12}A_{22} \\ -A_{21}A_{11} + A_{11}A_{21} & -A_{12}A_{21} + A_{11}A_{22} \end{bmatrix} . \quad \text{----- (2.6.4)}$$

By comparing the matrix elements in Eq.(2.6.4), therefore

$$\lambda (A_{22}A_{11} - A_{12}A_{21}) = 1 , \quad \text{----- (2.6.5)}$$

$$\lambda (A_{11}A_{21} - A_{21}A_{11}) = 0 , \quad \text{----- (2.6.6)}$$

$$\lambda (A_{22}A_{12} - A_{12}A_{22}) = 0 . \quad \text{----- (2.6.7)}$$

Eq.(2.6.5) can be written in another form thus

$$\lambda (A_{22}A_{11} - A_{12}A_{21}) = \lambda \det \mathbf{A} = 1 .$$

From Eqs.(2.6.6) and (2.6.7), we get

$$A_{11}A_{21} = A_{21}A_{11} ,$$

$$A_{22}A_{12} = A_{12}A_{22} , \quad \text{for } \lambda \neq 0$$

Since the matrix $\mathbf{A}$ is non-singular, we have

$$\lambda = \frac{1}{\det \mathbf{A}} \neq 0 .$$

The special unitary group SU(2) satisfies the above conditions, and thus the matrix $\mathbf{A}$ has unit determinant, i.e., $\det \mathbf{A} = 1$.

Now reconsider Eq.(2.6.5) $A_{11}A_{22} - A_{12}A_{21} = 1$. This can be satisfied in either of the two following cases only

case I $\quad A_{11}A_{22} = 1$ and $A_{12} = 0$,

case II $\quad A_{12}A_{21} = -1$ and $A_{11} = 0$.

Then, the matrices corresponding to the holomorphic transformations are

$$I_x = \begin{bmatrix} 0 & -i \\ -i & 0 \end{bmatrix}, \quad I_y = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}, \quad I_z = \begin{bmatrix} -i & 0 \\ 0 & i \end{bmatrix}.$$

For the spinor representation, the above matrices are infinitesimal generators of the rotation group

($R(\phi n) = \exp(i\phi n\sigma)$, for example $R(\pi x) = \begin{bmatrix} 0 & -i \\ -i & 0 \end{bmatrix}$, $R(\pi y) = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$, $R(\pi z) = \begin{bmatrix} -i & 0 \\ 0 & i \end{bmatrix}$). We have thus been able to correlate the rotations I_μ, where $\mu = x, y, z$, with the corresponding observables σ_μ (Pauli matrices). The infinitesimal generators will be written as $\frac{1}{2}\sigma$, that is,

$$R^{\frac{1}{2}}(\pi n_\mu) = \exp(-i\tfrac{1}{2}\pi\sigma_\mu) = I_\mu . \quad \text{-----} (2.6.8)$$

From the property $I_\mu^{\,2} = -I$, where I is the identity matrix, we find the relation :

$$\ln I_\mu = \tfrac{1}{2}\pi I_\mu \quad . \qquad \text{----- (2.6.9)}$$

We can verify Eq.(2.6.9) by expanding the anti-logarithm of Eq.(2.6.9), thus

$$\exp(\tfrac{1}{2}\pi I_\mu) = \sum \frac{(\tfrac{1}{2}\pi I_\mu)^n}{n!}$$

$$= I + \tfrac{1}{2}\pi I_\mu - \frac{1}{2!}(\tfrac{1}{2}\pi)^2 I - \frac{1}{3!}(\tfrac{1}{2}\pi)^3 I_\mu$$

$$+ \frac{1}{4!}(\tfrac{1}{2}\pi)^4 I + \frac{1}{5!}(\tfrac{1}{2}\pi)^5 I_\mu + \ldots$$

$$= I\cos(\tfrac{1}{2}\pi) + I_\mu \sin(\tfrac{1}{2}\pi)$$

$$= I_\mu \quad .$$

Therefore,

$$\exp(\tfrac{1}{2}\pi I_\mu) = I_\mu \quad . \qquad \text{----- (2.6.10)}$$

Comparing Eqs.(2.6.10) and (2.6.8), we get

$$\sigma_\mu = i I_\mu \quad . \qquad \text{----- (2.6.11)}$$

Eq.(2.6.11) is the relation of the bilateral binary rotations I_μ and σ_μ (μ = 1,2,3) represents the Pauli matrices. This confirms that the I_μ's are infinitesimal group generators for the spinor representations of SO(3).

The Pauli matrices σ_μ and the identity matrix I form a group. Physicists have used this group (I, σ_x, σ_y, σ_z) to generate spinors in the following way. In the case of the Pauli matrices, they can be rewritten as follows

$$\mathbf{A} = \begin{bmatrix} a & b \\ -b^* & a^* \end{bmatrix} \quad .$$

From Eq.(2.6.2), let us write ζ as the quotient of two new variables μ_1 and μ_2 . Thus

$$\frac{\bar{\mu}_1}{\bar{\mu}_2} = \frac{a\mu_1\mu_2^{-1} + b}{-b^*\mu_1\mu_2^{-1} + a^*} = \frac{a\mu_1 + b\mu_2}{-b^*\mu_1 + a^*\mu_2} \quad ,$$

or

$$|\,\bar{\mu}_1\bar{\mu}_1\,\rangle = \begin{bmatrix} a & b \\ -b^* & a^* \end{bmatrix} |\,\mu_1\mu_2\,\rangle \quad . \qquad \text{-----} \; (2.6.12)$$

A basis that transforms under (2.6.12) is called a *spinor* . The complex conjugate spinor is

$$|\,\bar{\mu}_1^*\bar{\mu}_2^*\,\rangle = \begin{bmatrix} a^* & b^* \\ -b & a \end{bmatrix} |\,\mu_1^*\mu_2^*\,\rangle \, . \qquad \text{-----} \; (2.6.13)$$

The matrices in (2.6.12) and (2.6.13) have the same trace, so that the conjugate spinor transforms under the same representation. It is easy to guess the transformation of the basis $|\mu_1^*\mu_2^*\rangle$ in (2.6.13) that will bring the matrix in (2.6.13) into precisely that in (2.6.12). Two-component *spinor calculus* is a very specific calculus for studying the structure of space-time manifolds. It is closely related to the calculus of the quaternions that represent a four-dimensional space with spin structure.

2.6.2 The Schrödinger and Dirac equations

The Schrödinger equation is one of the basic equations of nonrelativistic quantum mechanics. It describes the behaviour of a particle of mass m in a potential V in terms of the states of a quantum system. We can obtain the Schrödinger equation from the expression for the total energy E in classical mechanics :

$$E = \frac{p^2}{2m} + V \qquad \text{----- (2.6.14)}$$

by replacing the classical momentum P by the momentum operator

$$P \longrightarrow i\hbar \nabla , \qquad \text{----- (2.6.15)}$$

where $\nabla = \frac{\partial}{\partial x} \mathbf{i} + \frac{\partial}{\partial y} \mathbf{j} + \frac{\partial}{\partial z} \mathbf{k}$,

and the classical energy by the energy operator

$$E \longrightarrow i\hbar \frac{\partial}{\partial t} . \qquad \text{----- (2.6.16)}$$

The linear operators thus obtained from Eq.(2.6.14) act on the wave function $\psi(x,y,z,t)$:

$$\left(\frac{-\hbar^2\nabla^2}{2m} + V\right)\psi = i\hbar \frac{\partial\psi}{\partial t} ,$$

or

$$\frac{i\hbar\partial\psi}{\partial t} = H\psi , \qquad \text{----- (2.6.17)}$$

where $H = \frac{-\hbar^2\nabla^2}{2m} + V$, and H is called the *Hamiltonian operator*, or simply the *Hamiltonian*, of the system, which is the same as the energy operator. Eq.(2.6.17) is called the *Schrödinger wave equation*, and its solution is the time-dependent wave function.

The Schrödinger wave equation fails, however, to explain the behaviour of a particle in the *relativistic* quantum system. It is thus necessary to develop a proper relativistically covariant version of this equation, as follows. Consider first the relativistic energy-momentum relation

$$E^2 = (pc)^2 + (m_o c^2)^2 \quad , \qquad \text{-----} \; (2.6.18)$$

where E now includes the rest energy $m_o c^2$. By substituting the momentum operator (2.6.15) and the energy operator (2.6.16) in Eq.(2.6.18), and applying the resulting relativistic Hamiltonian operator on the wave function $\psi(x,y,z,t)$, we obtain the Klein-Gordon equation,

$$-\hbar^2 \frac{\partial^2 \psi}{\partial t^2} = -\hbar^2 c^2 \nabla^2 \psi + m_o^2 c^4 \psi \quad . \qquad \text{-----} \; (2.6.19)$$

This has plane wave solutions of the form

$$\exp\, i(\mathbf{kr} - \omega t) \quad , \qquad \text{-----} \; (2.6.20)$$

where

$$(\hbar\omega)^2 = \pm(\hbar^2 c k^2 + m_o^2 c^4)^{\frac{1}{2}} \quad . \qquad \text{-----} \; (2.6.21)$$

The Klein-Gordon equation (2.6.19) is not quite satisfactory for the description of the electron with spin $\frac{1}{2}$, since its solution can lead to negative probability which is physically unacceptable. In 1928, Dirac found a way out of this difficulty by substituting the relativistic Hamiltonian

$$H = c\alpha\cdot p + \beta m_o c^2 \qquad ----- (2.6.22)$$

into (2.6.17), which leads to the wave equation

$$(E - c\alpha\cdot p - \beta m_o c^2)\psi = 0 \quad ,$$

or $$----- (2.6.23)$$

$$(i\hbar\frac{\partial}{\partial t} + i\hbar c\alpha\cdot\nabla - \beta m_o c^2)\psi = 0 \quad ,$$

where $\alpha = \alpha_x + \alpha_y + \alpha_z$, α and β are anticommutative and satisfy the following relations

$$\alpha_x^2 = \alpha_y^2 = \alpha_z^2 = \beta^2 = I \quad ,$$

$$\alpha_x\alpha_y + \alpha_y\alpha_x = \alpha_y\alpha_z + \alpha_z\alpha_y = \alpha_z\alpha_x + \alpha_x\alpha_z = 0 \quad ,$$

$$\alpha_x\beta + \beta\alpha_x = \alpha_y\beta + \beta\alpha_y = \alpha_z\beta + \beta\alpha_z = 0 \ . \qquad ----- (2.6.24)$$

Any solution ψ of (2.6.23) is also a solution of (2.6.19), but the reverse may not be true in general. For the relativistic electron, we can thus avoid having physically unacceptable solutions. The α and β are not real, but can be expressed in terms of Hermitian 4x4 matrices. One representation of the α and β matrices is in terms of the familiar 2x2 Pauli matrices,

$$\sigma_x = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} , \quad \sigma_y = \begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix} , \quad \sigma_z = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} :$$

$$\alpha = \begin{bmatrix} 0 & \sigma \\ \sigma & 0 \end{bmatrix} , \quad \beta = \begin{bmatrix} I & 0 \\ 0 & -I \end{bmatrix} .$$

That is ,

$$\beta = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{bmatrix} , \quad \alpha_x = \begin{bmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix} ,$$

----- (2.6.25)

$$\alpha_y = \begin{bmatrix} 0 & 0 & 0 & -i \\ 0 & 0 & i & 0 \\ 0 & -i & 0 & 0 \\ i & 0 & 0 & 0 \end{bmatrix} , \quad \alpha_z = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \\ 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \end{bmatrix} .$$

The Dirac equation (2.6.22) for an electron, when written in the form

$$i\hbar\frac{\partial\psi}{\partial t} = c\left[(\hbar/i)\alpha\cdot\nabla + \beta m_o c\right]\psi , \qquad \text{----- (2.6.26)}$$

is the relativistic analogue of the time-dependent Schrödinger equation of nonrelativistic quantum mechanics. Its form is also rather similar to that of Maxwell's equations for the electromagnetic field. Multiplying Eq.(2.6.26) by β and dividing by $i\hbar c$, we get

$$\left[(\beta/c)\frac{\partial}{\partial t} + \beta\alpha\cdot\nabla\right]\psi + (im_o c\psi/\hbar) = 0 \quad . \qquad \text{----- (2.6.27)}$$

By using the relativistic notation of Eq.(2.6.27) and the summation convention, with Greek indices running from 0 to 3 , Eq.(2.6.27) may be rewritten in the compact form

$$\gamma^{\mu}\frac{\partial\psi}{\partial x^{\mu}} + iK\psi = 0 \quad , \qquad \text{----- (2.6.28)}$$

for μ = 0, 1, 2, 3 and $\gamma^{0} = \beta$, $\gamma^{1} = \beta\alpha_{x}$, $\gamma^{2} = \beta\alpha_{y}$, $\gamma^{3} = \beta\alpha_{z}$, $K = (m_{o}c/\hbar)$ and

$$\gamma^{\mu}\frac{\partial}{\partial x^{\mu}} = \gamma^{0}\frac{\partial}{\partial x^{0}} + \gamma\cdot\nabla$$

$$= \beta\left(\frac{1}{c}\right)\frac{\partial}{\partial t} + \beta\alpha\cdot\nabla \quad .$$

The Dirac matrices satisfy the relation

$$\gamma^{\mu}\gamma^{\nu} + \gamma^{\nu}\gamma^{\mu} = 2g^{\mu\nu}I \quad . \qquad \text{----- (2.6.29)}$$

The Dirac equation of the electron in an electromagnetic field is now written in the form

$$\gamma^{\mu}\left[\frac{\partial\psi}{\partial x^{\mu}} - \frac{ie}{\hbar c}A_{\mu}\right]\psi + iK\psi = 0 \quad , \qquad \text{----- (2.6.30)}$$

where $A_{\mu} = (\phi, \mathbf{A})$, and e is the electron charge. We use Eq.(2.6.30) to explain the motion of a free spin-$\frac{1}{2}$ particle in relativistic quantum theory, which includes the interaction with the electromagnetic field.

2.6.3 Twistors

Twistors are certain complex quantities which are holomorphically parametrised. They can also be considered as an extension of the concept of spinors. The geometries of the twistor theory are very useful in the mathematical analysis of the physical problems of quantization of fields and space-time structure. It is possible to use twistors to construct and explain physical concepts rather directly. The concepts of space-time points and curvature, of energy-momentum, angular momentum, quantization, the structure of elementary particles with their various internal quantum numbers, wave functions, space-time fields, can all be formulated, with varying degrees of speculativeness, completeness, and success, in a more or less direct way from primitive twistor concepts. The quantum field theory of elementary particles and their interactions in Minkowski space, and general relativity involve non-linear problems which can be conveniently expressed in the language of twistors and solved by new techniques of the twistor theory. The importance of complex numbers and complex analyticity in twistor geometry and analysis tells one something fundamental about the mathematical structure of the physical world. Excellent surveys of the twistor theory can be found in [35], [63] .

The basic twistor equation is

$$\nabla^{A}{}_{A'}\, \omega^{B} = 0 \quad , \qquad \text{----- (2.6.31)}$$

and the solutions are ω^A and $\pi_{A'}$. The solution ω^A is composed of four-dimensional vector spaces T^α over the complex numbers, called *twistor spaces*. The elements of a twistor space are called $\begin{bmatrix} 1 \\ 0 \end{bmatrix}$-*twistors*.

From these $\begin{bmatrix} 1 \\ 0 \end{bmatrix}$-twistors we can write

$$Z^\alpha = (\omega^A, \pi_{A'}) \quad , \qquad \text{----- (2.6.32)}$$

where ω^A and $\pi_{A'}$ could be either spinor fields or point spinors at the spinor frame 0 and called *spinor parts*. Multiplication by a complex number and addition of twistors are defined in an obvious way :

$$\lambda (\omega^A, \pi_{A'}) = (\lambda\omega^A, \lambda\pi_{A'}) \quad ,$$

$$\text{----- (2.6.33)}$$

$$(\omega^A, \pi_{A'}) + (\xi^A, \eta_{A'}) = (\omega^A + \xi^A, \pi_{A'} + \eta_{A'}) \quad .$$

Twistors can thus be built up by using spinor parts.

The $\begin{bmatrix} 1 \\ 0 \end{bmatrix}$-*twistor space* is effectively the direct sum of the space spinors of type ω^A and $\pi_{A'}$ at the spinor frame 0. Similarly, the dual of $\begin{bmatrix} 1 \\ 0 \end{bmatrix}$-twistor space, that is $\begin{bmatrix} 0 \\ 1 \end{bmatrix}$-twistor space, is the sum of spinors of type λ_A, $\mu^{A'}$ at O. We can write

$$\omega_\alpha \longleftrightarrow (\lambda_A(o), \mu^{A'}(o)) \quad . \qquad \text{----- (2.6.34)}$$

The product of $\begin{bmatrix} 1 \\ 0 \end{bmatrix}$-twistor and its dual is

$$\omega_\alpha Z^\alpha = (\lambda_A(o), \mu^{A'}(o))(\omega^A(o), \pi_{A'}(o))$$

----- (2.6.35)

$$= (\lambda_A \omega^A(o), \mu^{A'} \pi_{A'}(o)) \quad .$$

We will now consider the *higher-valence* twistors. The outer product $X^\alpha Z^\beta$ of two twistors such as

$$X^\alpha = (\xi^A, \eta_{A'}) \quad \text{and} \quad Z^\alpha = (\omega^A, \pi_{A'}) \quad ,$$

represented at O by

$$X^\alpha \overset{o}{\longleftrightarrow} (\xi^A(o), \eta_{A'}(o)) \quad \text{and} \quad Z^\alpha \overset{o}{\longleftrightarrow} (\omega^A(o), \pi_{A'}(o)) \quad ,$$

----- (2.6.36)

is

$$X^\alpha Z^\beta = (\xi^A, \eta_{A'})(\omega^B, \pi_{B'})$$

----- (2.6.37)

$$= (\xi^A \omega^B, \xi^A \pi_{B'}, \eta_{A'} \omega^B, \eta_{A'} \pi_{B'}) \quad ,$$

where the components of $X^\alpha Z^\beta$ consist of four spinors. An important feature of the sum of products of type $X^\alpha Z^\beta$ is the general $\begin{bmatrix} 2 \\ 0 \end{bmatrix}$-*twistor* $S^{\alpha\beta}$:

$$S^{AB} , \quad S^{A}{}_{B'} , \quad S_{A'}{}^{B} , \quad S_{A'B'} . \qquad ----- (2.6.38)$$

It is often convenient to introduce separate symbols for the various spinor parts. The notation of the twistors is

$$S^{\alpha\beta} = \begin{bmatrix} S^{AB} & S^{A}{}_{B'} \\ S_{A'}{}^{B} & S_{A'B'} \end{bmatrix} .$$

In the same way we write the patterns of $\begin{bmatrix} 1 \\ 1 \end{bmatrix}$- and $\begin{bmatrix} 0 \\ 2 \end{bmatrix}$-twistors as :

$$E^{\alpha}{}_{\beta} = \begin{bmatrix} E^{A}{}_{B} & E^{AB'} \\ E_{A'B} & E_{A'}{}^{B'} \end{bmatrix} , \quad R_{\alpha\beta} = \begin{bmatrix} R_{AB} & R_{A}{}^{B'} \\ R^{A'}{}_{B} & R^{A'B'} \end{bmatrix} .$$

These are 2^{p+q} spinor parts of the $\begin{bmatrix} p \\ q \end{bmatrix}$-twistor because of the spin up and down. Another example of a $\begin{bmatrix} 2 \\ 1 \end{bmatrix}$-twistor $T^{\alpha\beta}{}_{\gamma}$ which has eight independent spinor parts is

$$T^{AB}{}_{C} , \quad T_{A'}{}^{B}{}_{C} , \quad T^{ABC'} , \quad T_{A'}{}^{BC'} ,$$
$$T^{A}{}_{B'C} , \quad T_{A'B'C} , \quad T^{A}{}_{B'}{}^{C'} , \quad T_{A'B'}{}^{C'} .$$

The specific notation of $\begin{bmatrix} 2 \\ 0 \end{bmatrix}$- and $\begin{bmatrix} 1 \\ 1 \end{bmatrix}$-twistors are

$$S^{\alpha\beta} = \begin{bmatrix} \sigma^{AB} & \rho^{A}{}_{B'} \\ \tau_{A'}{}^{B} & \kappa_{A'B'} \end{bmatrix} , \quad E^{\alpha}{}_{\beta} = \begin{bmatrix} \theta^{A}{}_{B} & \xi^{AB'} \\ \eta_{A'B} & \zeta_{A'}{}^{B'} \end{bmatrix} .$$

CHAPTER III

GENERALIZED VECTOR MATRICES, BIMATRICES, AND OCTONIONS

The *octonions* and *Zorn's vector matrices* have eight-dimensional structures. Some might wonder how they really "look like". In Section 1 the algebraic structures of octonions and Zorn's vector matrices are considered. This leads naturally to the topic of "graded Lie-admissibility of vector matrix algebras", which is outlined in Section 2. *Bimatrices*, which are closely related to the special case of $\mathbf{Z}_2$-grading, are then introduced in Section 3. In the last section, we describe the power-associative products on octonions. The graphical representations of the vector matrices and their different types of product will be considered in Chapter IV. Some of the ideas and results of these two chapters will then be applied to the problems of mechanics, robotics, and optics, in Chapter V.

Section 3.1 Octonions and Zorn's Vector Matrices

In this section we will consider the *octonions* or *Cayley numbers*, which form an eight-dimensional vector space. Their products are, in general, both *nonassociative* and *noncommutative*. We will introduce the "split octonions", which give an algebraic structure closely related to that of the octonions. It will be shown how the split octonions can be generated from the octonions, and how they are related to Zorn's vector matrices. ([3], [5],[8],[22],[23],[30],[32],[59])

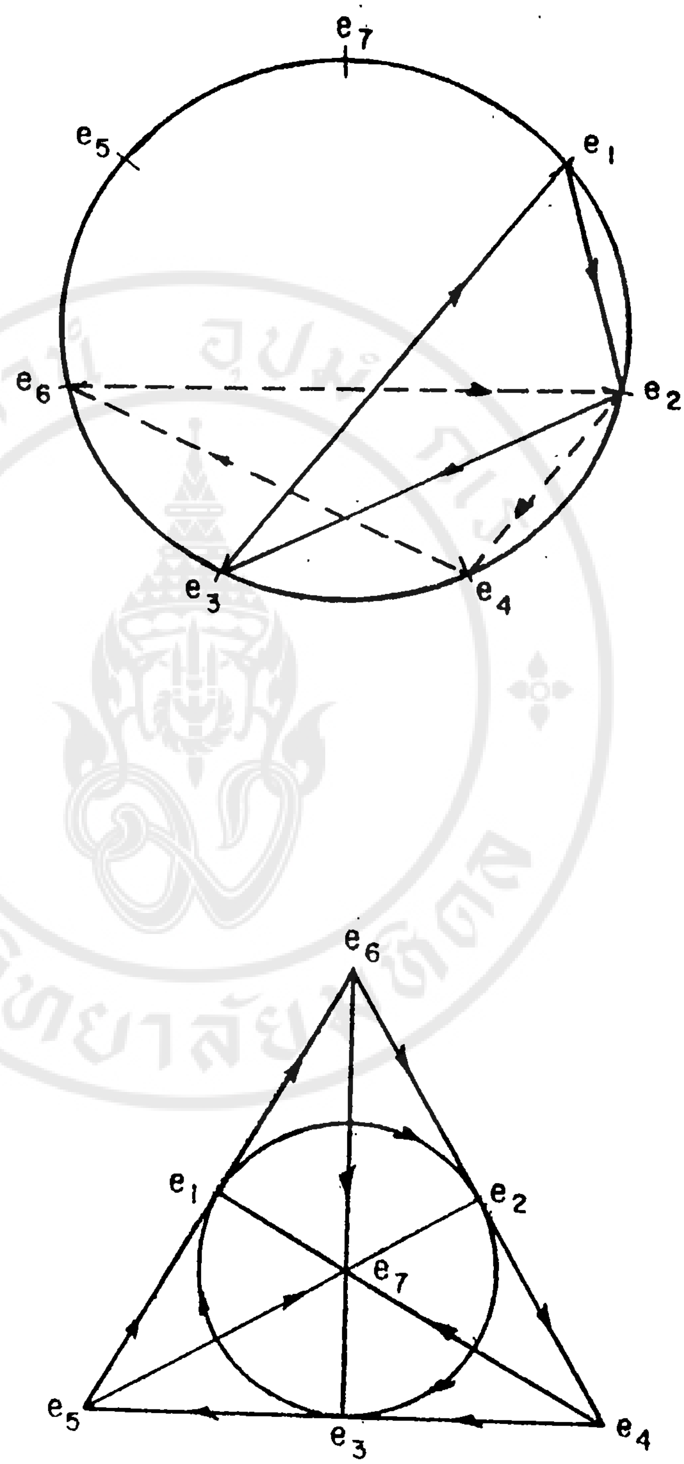
3.1.1 The octonion algebra or Cayley algebra

A *composition algebra* is defined as an algebra A with identity and with a nondegenerate quadratic form Q defined over it such that Q permits composition, i.e., for $x,y \in A$,

$$Q(xy) = Q(x)Q(y) . \qquad ----- (3.1.1)$$

There exist composition algebras over the real and complex number fields, over the skew-field of quaternions, and over the nonassociative division ring of octonions. Their dimensions are, respectively, 1, 2, 4, and 8 . The multiplication of octonions is neither commutative nor associative. A composition algebra is said to be a *division algebra* if the quadratic form Q is anisotropic i.e., if $Q(x) = 0$ implies that $x = 0$, otherwise the algebra is called *split*. A split algebra may have *zero-divisors*. A basis for the real octonion $\mathbb{O}$ has eight elements, including the identity $\mathbf{1}$, and the imaginary units e_A $(A = 1,\ldots,7)$, where $e_A{}^2 = -1$. These basis elements satisfy the following multiplication table (Table 3.1A). The multiplication rules of the seven imaginary units of octonions can be conveniently summarized in Fig. 3.1(a) and (b) .

(a)



(b)

Fig. 3.1 The seven imaginary units of octonions

Table 3.1A The multiplication table of octonionic units

*	1	e_1	e_2	e_3	e_4	e_5	e_6	e_7
1	1	e_1	e_2	e_3	e_4	e_5	e_6	e_7
e_1	e_1	-1	e_3	$-e_2$	e_7	$-e_6$	e_5	$-e_4$
e_2	e_2	$-e_3$	-1	e_1	e_6	e_7	$-e_4$	$-e_5$
e_3	e_3	e_2	$-e_1$	-1	$-e_5$	e_4	e_7	$-e_6$
e_4	e_4	$-e_7$	$-e_6$	e_5	-1	$-e_3$	e_2	e_1
e_5	e_5	e_6	$-e_7$	$-e_4$	e_3	-1	$-e_1$	e_2
e_6	e_6	$-e_5$	e_4	$-e_7$	$-e_2$	e_1	-1	e_3
e_7	e_7	e_4	e_5	e_6	$-e_1$	$-e_2$	$-e_3$	-1

It can be seen that

$$e_A e_B + e_B e_A = - 2 \delta_{AB} \; ; \qquad \text{-----} \; (3.1.2)$$

more concisely, $e_A e_B = \varepsilon_{ABC} \, e_C - \delta_{AB}$ ----- (3.1.3) where δ_{AB} is the Kronecker delta,

and ε_{ABC} is the totally antisymmetric Levi-Civita symbol:

$$\varepsilon_{ABC} = \varepsilon_{BCA} = \varepsilon_{CAB} = +1 \qquad \text{----- (3.1.4)}$$

and

$$\varepsilon_{BAC} = \varepsilon_{ACB} = \varepsilon_{CBA} = -1 \qquad \text{----- (3.1.5)}$$

for $ABC \in \{ 123,147,165,246,257,354,367 \}$.

Note here the cyclic symmetry obtained by ordering seven points clockwise on a circle with numbering (1 2 4 3 6 5 7), as given in Fig. 3.1(a). Then a triangle ABC is obtained from (123) by 6 successive rotations of angle $2\pi/7$. In Fig.3.1(a) the elements corresponding to the corners of the triangle form a basis of a quaternion subalgebra. Another convenient way of representing the multiplication table of the imaginary units of octonions is given in Fig. 3.1(b).

From the above multiplication table, it is clear that the algebra $\mathbb{O}$ is *not* associative. Yet it satisfies a weaker condition than associativity, namely *alternativity*. The *associator* $[x,y,z]$ of the elements x,y,z is defined as

$$[x,y,z] = [xy]z - x[yz] \,. \qquad \text{----- (3.1.6)}$$

The condition for alternativity requires the associator $[x,y,z]$ to be an alternating function of x,y,z :

$$[x,y,z] = [z,x,y] = [y,z,x] \qquad \text{----- (3.1.7)}$$

$$= -[y,x,z] = -[x,z,y] = -[z,y,x] \,.$$

The *octonion algebra* $\mathbb{O}$ with the above basis, considered over the real numbers $\mathbb{R}$, is a division algebra with the *quadratic form* Q defined by

$$Q(x) = x\bar{x} = \bar{x}x \,, \qquad \text{----- (3.1.8)}$$

where $\bar{x}$ is the octonion conjugate of x obtained by replacing e_A in x by $-e_A$

$$x = x_o + x_A e_A \,, \quad \bar{x} = x_o - x_A e_A \,. \qquad \text{----- (3.1.9)}$$

This quadratic form is also called the *norm form* and denoted by N(x). Then

$$N(x) = x\bar{x} = \bar{x}x = x_o^2 + \Sigma\, x_A^2 \,. \qquad \text{----- (3.1.10)}$$

3.1.2 The split octonions

For the *split* octonion algebra we choose the following basis

$$U_o = \frac{1}{2}(1 + ie_7) \,, \qquad U_o^* = \frac{1}{2}(1 - ie_7) \,,$$

$$U_1 = \frac{1}{2}(e_1 + ie_4) \,, \qquad U_1^* = \frac{1}{2}(e_1 - ie_4) \,,$$

$$\text{----- (3.1.11)}$$

$$U_2 = \frac{1}{2}(e_2 + ie_5) \,, \qquad U_2^* = \frac{1}{2}(e_2 - ie_5) \,,$$

$$U_3 = \frac{1}{2}(e_3 + ie_6) \,, \qquad U_3^* = \frac{1}{2}(e_3 - ie_6) \,,$$

where $i = \sqrt{-1}$ and is assumed to commute with all e_A. These basis elements satisfy the multiplication table :

$$U_i U_j = \varepsilon_{ijk} U_k^{*} \quad , \quad U_i^{*} U_j^{*} = \varepsilon_{ijk} U_k \quad ,$$
$$U_i U_j^{*} = -\delta_{ij} U_o \quad , \quad U_i^{*} U_j = -\delta_{ij} U_o^{*} \quad ,$$
$$U_i U_o^{*} = U_i \quad , \quad U_i^{*} U_o = U_i^{*} \quad ,$$

----- (3.1.12)

$$U_o U_i = U_i \quad , \quad U_o^{*} U_i^{*} = U_i^{*} \quad ,$$
$$U_o^{2} = U_o \quad , \quad U_o^{*2} = U_o^{*} \quad ,$$
$$U_i U_o = U_i^{*} U_o^{*} = U_o^{*} U_i = U_o U_i^{*} = U_o U_o^{*} = U_o^{*} U_o = 0 \quad ,$$

$i,j,k = 1,2,3$.

Table 3.1B The multiplication table of split octonions

*	U_o	U_j	U_o^{*}	U_j^{*}
U_o	U_o	U_j	0	0
U_i	0	$\varepsilon_{ijk} U_k^{*}$	U_i	$-\delta_{ij} U_o$
U_o^{*}	0	0	U_o^{*}	U_j^{*}
U_i^{*}	U_i^{*}	$-\delta_{ij} U_o^{*}$	0	$\varepsilon_{ijk} U_k$

(a)

$i,j,k = 1,2,3$.

or

*	U_0	U_1	U_2	U_3	U_0^*	U_1^*	U_2^*	U_3^*
U_0	U_0	U_1	U_2	U_3	0	0	0	0
U_1	0	0	U_3^*	$-U_2^*$	U_1	$-U_0$	0	0
U_2	0	$-U_3^*$	0	U_1^*	U_2	0	$-U_0$	0
U_3	0	U_2^*	$-U_1^*$	0	U_3	0	0	$-U_0$
U_0^*	0	0	0	0	U_0^*	U_1^*	U_2^*	U_3^*
U_1^*	U_1^*	$-U_0^*$	0	0	0	0	U_3	$-U_2$
U_2^*	U_2^*	0	$-U_0^*$	0	0	$-U_3$	0	U_1
U_3^*	U_3^*	0	0	$-U_0^*$	0	U_2	$-U_1$	0

(b)

Clearly, the split octonion algebra contains zero-divisors, hence it is not a division algebra.

We can select any group of the sets { 123 , 147 , 165 , 246 , 257 , 354 , 367 } to form the basis of split octonions, for example,

$$V_0 = \frac{1}{2}(1 + ie_1) \ , \qquad V_0^* = \frac{1}{2}(1 - ie_1) \ ,$$

$$V_1 = \frac{1}{2}(e_2 + ie_3) \ , \qquad V_1^* = \frac{1}{2}(e_2 - ie_3) \ ,$$

$$V_2 = \frac{1}{2}(e_4 + ie_7) \ , \qquad V_2^* = \frac{1}{2}(e_4 - ie_7) \ ,$$

$$V_3 = \frac{1}{2}(e_6 + ie_5) \ , \qquad V_3^* = \frac{1}{2}(e_6 - ie_5) \ ,$$

or

----- (3.1.13)

$$W_0 = \frac{1}{2}(1 + ie_2) \ , \qquad W_0^* = \frac{1}{2}(1 - ie_2) \ ,$$

$$W_1 = \frac{1}{2}(e_3 + ie_1) \ , \qquad W_1^* = \frac{1}{2}(e_3 - ie_1) \ ,$$

$$W_2 = \frac{1}{2}(e_4 + ie_6) \ , \qquad W_2^* = \frac{1}{2}(e_4 - ie_6) \ ,$$

$$W_3 = \frac{1}{2}(e_5 + ie_7) \ , \qquad W_3^* = \frac{1}{2}(e_5 - ie_7) \ .$$

The multiplication tables of the above two groups of basis are quite similar to Table 3.1B .

3.1.3 Zorn's vector matrices

A realization of the split octonion algebra is via the *Zorn's vector matrices*

$$\begin{bmatrix} a & \mathbf{X} \\ \mathbf{Y} & b \end{bmatrix} \ , \quad \text{where } a \text{ and } b \text{ are scalars,}$$

and **X** and **Y** are three-dimensional vectors. We can relate the split octonions to the vector matrices; namely

$$U_0 = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} , \qquad U_0^* = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} ,$$

----- (3.1.14)

$$U_i = \begin{bmatrix} 0 & e_i \\ 0 & 0 \end{bmatrix} , \qquad U_i^* = \begin{bmatrix} 0 & 0 \\ e_i & 0 \end{bmatrix} ;$$

then, $A = aU_0 + x_i U_i + bU_0^* + y_i U_i^*$. ----- (3.1.15)

If $A = \begin{bmatrix} a & X \\ Y & b \end{bmatrix}$ and $B = \begin{bmatrix} c & W \\ Z & d \end{bmatrix}$,

----- (3.1.16)

then the multiplication table of AB is

Table 3.1C The multiplication of split octonions

A \ B	U_0	U_j	U_0^*	U_j^*
U_0	acU_0	aW_jU_j	0	0
U_i	0	$X_iW_j\varepsilon_{ijk}U_k^*$	dX_iU_i	$-X_iZ_j\delta_{ij}U_0$
U_0^*	0	0	bdU_0^*	$bZ_jU_j^*$
U_i^*	$cY_iU_i^*$	$-Y_iW_j\delta_{ij}U_0^*$	0	$Y_iZ_j\varepsilon_{ijk}U_k$

$i, j, k = 1, 2, 3$.

Thus

$$\begin{bmatrix} a & X \\ Y & b \end{bmatrix} \begin{bmatrix} c & W \\ Z & d \end{bmatrix} = \begin{bmatrix} ac - X \cdot Z & aW + dX + Y \wedge Z \\ cY + bZ + X \wedge W & bd - Y \cdot W \end{bmatrix} , \quad \text{----- (3.1.17)}$$

where $\cdot$ and $\wedge$ denote the usual dot and vector product, respectively, and $e_i \wedge e_j = \varepsilon_{ijk} e_k$, and $e_i \cdot e_j = \delta_{ij}$, for $i,j,k = 1,2,3$.

Zorn's vector matrices satisfy all the conditions of the split octonion algebra with the product stated above. Octonion conjugation as defined above induces a natural *involution* for the vector matrices, as follows. Let

$$A = \begin{bmatrix} a & X \\ Y & b \end{bmatrix} , \quad \bar{A} = \begin{bmatrix} b & -X \\ -Y & a \end{bmatrix} , \quad \text{----- (3.1.18)}$$

$$A = aU_o + x_i U_i + bU_o^* + y_i U_i^* ,$$

$$\bar{A} = bU_o - x_i U_i + aU_o^* - y_i U_i^* .$$

Then $A\bar{A} = \bar{A}A = \begin{bmatrix} a & X \\ Y & b \end{bmatrix} \begin{bmatrix} b & -X \\ -Y & a \end{bmatrix}$

$$= \begin{bmatrix} ab + X \cdot Y & 0 \\ 0 & ab + X \cdot Y \end{bmatrix} . \quad \text{----- (3.1.19)}$$

Therefore $N(A) = \sqrt{(ab + X \cdot Y)(ab + X \cdot Y)}$, ----- (3.1.20)

$$\text{Norm}(A) = ab + X \cdot Y . \quad \text{----- (3.1.21)}$$

Section 3.2 Graded Lie-Admissibility of Vector Matrix Algebras

The purpose of this section is to study vector matrices in relation to the graded Lie-admissibility of superalgebras, which is the idea of Anargyros G. Fellouris . A special case of Myung product is considered in the case of superalgebras.

3.2.1 $\mathbb{Z}_2$-graded algebras

DEFINITION : An algebra A is said to be a *$\mathbb{Z}_2$-graded algebra* if $A = A_0 \oplus A_1$, with $A_i \oplus A_j \subset A_{i+j \pmod 2}$ for any $i,j = 0,1$. A *morphism* of $\mathbb{Z}_2$-graded algebras is a homomorphism $\phi : A \longrightarrow B$ such that $\phi(A_i) \subset B_i$ for any $i = 0,1$.

A *$\mathbb{Z}_2$-graded vector space* V , $\mathbb{Z}_2 = \{0, 1\}$, is the direct sum of two subspaces : V_0 , the even subspace, and V_1 , the odd subspace. The elements of the even (respectively, odd) subspace are called even (respectively, odd) . If A is any graded algebra, then the elements of A_i are said to be homogeneous of degree i . Therefore, both even and odd elements are said to be homogeneous. On the set $V_0 \cup V_1$ of all homogeneous elements, we define a sign function σ by

$$\sigma(x) = \begin{cases} 0 , & \text{if } x \text{ is even} , \\ 1 , & \text{if } x \text{ is odd} . \end{cases}$$

3.2.2 A graded Lie-admissible superalgebra

For any algebra A, denote by A^- the algebra with multiplication $[x,y] = xy - yx$ defined on the vector space A. We call A *Lie-admissible* if A^- is a Lie algebra, that is, if A^- satifies the Jacobi identity.

A *Lie superalgebra* M is a superalgebra over a field F whose multiplication, denoted by a bracket [,], satifies the following conditions:

$$[x,y] = -(-1)^{d(x)d(y)}[y,x] \quad , \qquad ----- (3.2.1)$$

for all homogeneous x,y,z in M (graded skew symmetry),

$$0 = (-1)^{d(x)d(z)}[[x,y],z] + (-1)^{d(y)d(x)}[[y,z],x] + (-1)^{d(z)d(y)}[[z,x],y] \quad , \qquad ----- (3.2.2)$$

for all homogeneous x, y, z in M (graded Jacobi identity). We note that the even part M_o of a Lie superalgebra M is an ordinary Lie algebra.

Let $A = A_o \oplus A_1$ be a superalgebra whose multiplication is denoted by the bracket [,] on A :

$$[x,y] = xy - (-1)^{d(x)d(y)}yx \qquad ----- (3.2.3)$$

for all homogeneous elements x,y in A, and we extend it by linearity to all A. When the superalgebra A, endowed with the multiplication

[,], becomes a Lie superalgebra, A is called a *graded Lie-admissible superalgebra*.

Clearly, every associative superalgebra is a graded Lie-admissible superalgebra.

Using the *associator* defined by

$$[x,y,z] = (xy)z - x(yz) , \qquad \text{----- (3.2.4)}$$

we call a superalgebra A *superflexible* if it satisfies

$$(z,y,x) = -(-1)^{\Sigma(x,y,z)}(x,y,z) \qquad \text{----- (3.2.5)}$$

for all homogeneous elements x,y,z in A, where

$$\Sigma(x,y,z) = \sigma(x)\sigma(y) + \sigma(y)\sigma(z) + \sigma(z)\sigma(x) . \qquad \text{----- (3.2.6)}$$

A graded Lie-admissible, superalgebra A which satisfies the superflexible law (3.2.5) is called a *superflexible graded Lie-admissible superalgebra*.

A superalgebra A is *superflexible graded Lie-admissible* if and only if the graded derivative law :

$$[x,yz] = [x,y]z + (-1)^{\sigma(x)\sigma(y)}y[x,z] \qquad \text{----- (3.2.7)}$$

is satisfied by all homogeneous elements x,y,z in A.

3.2.3 The 2 x 2 matrix algebra M

Let L be an anticommutative algebra over a field F ; that is, L satisfies the anticommutative law $xy = -yx$, $x,y \in L$. Suppose that L possesses a symmetric bilinear form $[\ ,\ ]$ satisfying the invariant condition

$$(xy,z) = (x,yz)\ , \qquad x,y,z \in L\ . \qquad \text{------ } (3.2.8)$$

We consider the set of 2 x 2 matrices with diagonal entries in F and off-diagonal entries in L , defined by

$$M = \left\{ \begin{bmatrix} a & X \\ Y & b \end{bmatrix} \;\middle|\; a,b \in F\ ,\ x,y \in L \right\} ;$$

i.e., $M = M(L,s,t)$, where s and t are scalars. We define multiplication in M by

$$\begin{bmatrix} a & X \\ Y & b \end{bmatrix} \begin{bmatrix} c & U \\ V & d \end{bmatrix} = \begin{bmatrix} ac + (X,V) & aU + dX + tYV \\ cY + bV + sXU & bd + (Y,U) \end{bmatrix} .$$

$$\text{-----} \ (3.2.9)$$

M is a *flexible quadratic algebra* . That is, M is an algebra over F with unit element $I = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$, which satisfies the flexible law

$$x(yx) = (xy)x \ , \qquad \text{-----} \ (3.2.10)$$

for all $x,y,z \in M$, and a quadratic equation

$$A^2 - (a + b)A + (ab - (x,y)) = 0 \qquad \text{----- (3.2.11)}$$

for all $A = \begin{bmatrix} a & X \\ Y & b \end{bmatrix} \in M$.

Since in Eq.(3.2.11) A^2 is a linear combination of I and A, it follows from Eq.(3.2.10) that M satisfies the Jordan identity

$$A^2(BA) = (A^2B)A \quad \text{for all } A,B \in M. \qquad \text{----- (3.2.12)}$$

If a flexible algebra satisfies Eq.(3.2.12), then they are called a *noncommutative Jordan algebra*.

Let L be the three-dimensional simple Lie algebra with the outer product as multiplication, and let (,) be the usual inner product. Then the algebra M (L,1,-1) with s = 1, t = -1 defined by Eq.(3.2.9) is the *split octonion algebra* (char F ≠ 2). If L is a three-dimensional vector space in which the outer product is the vector product and the inner product is the dot product then the multiplication is

$$\begin{bmatrix} a & X \\ Y & b \end{bmatrix}\begin{bmatrix} c & U \\ V & d \end{bmatrix} = \begin{bmatrix} ac + (X\cdot V) & aU + dX - Y\wedge V \\ cY + bV + X\wedge U & bd + (Y\cdot U) \end{bmatrix}.$$

----- (3.2.13)

This multiplication is the previous multiplication of Zorn's vector matrices.

Let L be an algebra over F with a symmetric bilinear form $(\ ,\ )$, and A be an algebra over F with a symmetric linear form ϕ (i.e., $\phi(ab) = \phi(ba)$, $a, b \in A$). Denote by $M_e(L,A,\phi)$ the set of 2x2 matrices $\begin{bmatrix} a & X \\ Y & b \end{bmatrix}$, $X,Y \in L$, $a,b \in A$.

We define a multiplication in $M_e(L,A,\phi)$ as

$$\begin{bmatrix} a & X \\ Y & b \end{bmatrix} \begin{bmatrix} c & U \\ V & d \end{bmatrix} = \begin{bmatrix} ac + (X,V)e & \phi(a)U+\phi(d)X+\phi(f)YV \\ \phi(c)Y+\phi(b)V+\phi(g)XU & bd + (Y,U)e \end{bmatrix}, \quad \text{----- (3.2.14)}$$

where e, f and g are fixed in A.

Anargyros G. Fellouris [25] used this matrix to find the multiplication of superalgebra, that is,

$$\begin{bmatrix} a & X \\ Y & b \end{bmatrix} \begin{bmatrix} c & U \\ V & d \end{bmatrix} = \begin{bmatrix} ac + (X,V)e & \phi(a)U + \phi(d)X + tYV \\ \phi(c)Y + \phi(b)V + sXU & bd + (Y,U)e \end{bmatrix}, \quad \text{----- (3.2.15)}$$

where e is fixed in A, and s,t are scalars in F.

Consider the subspace of m as

$$M_o = \left\{ \begin{bmatrix} a & 0 \\ 0 & b \end{bmatrix} \mid a,b \in A \right\},$$

$$M_1 = \left\{ \begin{bmatrix} 0 & X \\ Y & 0 \end{bmatrix} \mid X,Y \in L \right\}.$$

Clearly M is the direct sum of M_o and M_1, but in order that M possesses the structure of a *graded* algebra the following inclusion relations must hold :

$$M_oM_o \subseteq M_o \quad , \qquad M_oM_1 \subseteq M_1 \quad , \quad \text{and}$$
$$M_1M_1 \subseteq M_o \quad . \qquad\qquad ----- (3.2.16)$$

Consider the product M_1M_1 from two arbitrary odd elements

$$A_1 = \begin{bmatrix} 0 & \mathbf{X} \\ \mathbf{Y} & 0 \end{bmatrix} \quad \text{and} \quad B_1 = \begin{bmatrix} 0 & \mathbf{U} \\ \mathbf{V} & 0 \end{bmatrix}, \qquad ----- (3.2.17)$$

viz. $$A_1B_1 = \begin{bmatrix} (\mathbf{X},\mathbf{V})e & t\mathbf{Y}\mathbf{V} \\ s\mathbf{X}\mathbf{U} & (\mathbf{Y},\mathbf{U})e \end{bmatrix} . \qquad ----- (3.2.18)$$

If AB satisfies Eq.(3.2.16), then t and s are both zero. Eq.(3.2.15) becomes

$$\begin{bmatrix} a & \mathbf{X} \\ \mathbf{Y} & b \end{bmatrix} \begin{bmatrix} c & \mathbf{U} \\ \mathbf{V} & d \end{bmatrix} = \begin{bmatrix} ac + (\mathbf{X},\mathbf{V})e & \phi(a)\mathbf{U} + \phi(d)\mathbf{X} \\ \phi(c)\mathbf{Y} + \phi(b)\mathbf{V} & bd + (\mathbf{Y},\mathbf{U})e \end{bmatrix} .$$

----- (3.2.19)

3.2.4 The graded Lie-admissibility of M

We now investigate the *graded Lie-admissibility* of the nonassociative superalgebra M defined in Section 3.2.3 . Let

$$A_o = \begin{bmatrix} a & 0 \\ 0 & b \end{bmatrix} , \qquad B_o = \begin{bmatrix} c & 0 \\ 0 & d \end{bmatrix}$$

----- (3.2.20)

be two elements in M_0, and A_1, B_1 be the two arbitrary odd elements given by Eq.(3.2.17). Then, using the definition (3.2.3) and (3.2.19), we find

$$[A_0,B_0] = \begin{bmatrix} ac - ca & 0 \\ 0 & bd - db \end{bmatrix},$$

$$[A_0,A_1] = \begin{bmatrix} 0 & [\phi(a) - \phi(b)]X \\ [\phi(b) - \phi(a)]Y & 0 \end{bmatrix},$$

$$[A_1,B_1] = \begin{bmatrix} [(X,V) + (U,Y)]e & 0 \\ 0 & [(Y,U) + (V,X)]e \end{bmatrix}.$$

Hence the inclusion relations

$$[M_0,M_0] \subseteq M_0 , \quad [M_0,M_1] \subseteq M_1 , \quad [M_1,M_1] \subseteq M_0 \qquad ----- (3.2.21)$$

are clearly satisfied. The condition of graded commutativity is also satisfied, because

$$\begin{aligned} [B,C] &= BC - (-1)^{d(B)d(C)}CB \\ &= -(-1)^{d(B)d(C)}[C,B] \end{aligned}$$

for all homogeneous elements B, C in $M_0 \cup M_1$. The graded nonassociative superalgebra M, defined by

$$M = \left\{ \begin{bmatrix} a & X \\ Y & b \end{bmatrix} \mid a,b \in A ; X,Y \in L \right\},$$

$$M_o = \{ \begin{bmatrix} a & 0 \\ 0 & b \end{bmatrix} \mid a,b \in A \} ,$$

$$M_1 = \{ \begin{bmatrix} 0 & X \\ Y & 0 \end{bmatrix} \mid X,Y \in L \} ,$$

that satisfies Eq.(3.2.15), is a graded Lie-admissible superalgebra provided that the following conditions are satisfied :

(i) $s = t = 0$.

(ii) The algebra A is commutative or associative or a Lie algebra.

(iii) The linear form ϕ on A is symmetric.

(iV) e belongs to the center of A .

(V) The bilinear form (,) on L is symmetric.

3.2.5 The graded Lie-admissibility of vector matrix algebras

Consider the vector matrix

$$M = \begin{bmatrix} a & X \\ Y & b \end{bmatrix} .$$

Specify

$$U_o = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} , \quad U_o^* = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} ,$$

$$U_i = \begin{bmatrix} 0 & e_i \\ 0 & 0 \end{bmatrix} , \quad U_i^* = \begin{bmatrix} 0 & 0 \\ e_i & 0 \end{bmatrix} ,$$

for $i = 1, 2, 3$.

Let $M_o^* = aU_o + bU_o^* = \begin{bmatrix} a & 0 \\ 0 & b \end{bmatrix}$,

and $M_1^* = \Sigma(X_i U_i + Y_i U_i^*) = \begin{bmatrix} 0 & X \\ Y & 0 \end{bmatrix}$,

$X , Y \in \mathbb{R}^3$, and let $A_o^*, B_o^* \in M_o^*$,

where $A_o^* = \begin{bmatrix} a & 0 \\ 0 & b \end{bmatrix}$, $B_o^* = \begin{bmatrix} c & 0 \\ 0 & d \end{bmatrix}$,

and $A_1^*, B_1^* \in M_1^*$,

where $A_1^* = \begin{bmatrix} 0 & X \\ Y & 0 \end{bmatrix}$, $B_1^* = \begin{bmatrix} 0 & U \\ V & 0 \end{bmatrix}$.

Using the definition (3.2.0) and (3.2.9) again, we find that

$$[M_o^*, M_o^*] = \{ 0 \} ,$$
$$[M_o^*, M_1^*] = M_1^* , \qquad \text{----- (3.2.22)}$$
$$[M_1^*, M_1^*] = M_o^* .$$

Then superflexible graded Lie-admissible superalgebra of vector matrices becomes trivial, because of the relation $[M_o^*, M_o^*] = \{ 0 \}$.

Section 3.3 Bimatrices

In Section 3.1 the multiplication table of octonions is shown in terms of the basis $(1, e_1, e_2, \ldots, e_7)$. Now we will introduce the representative matrices of the units of this basis by using the forms of the i-type and ε-type hypernumbers, namely,

$$i_0 = \begin{bmatrix} i & 0 \\ 0 & i \end{bmatrix}, \quad i_1 = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix},$$

$$i_2 = \begin{bmatrix} i & 0 \\ 0 & -i \end{bmatrix}, \quad i_3 = \begin{bmatrix} 0 & i \\ i & 0 \end{bmatrix}, \qquad \text{----- (3.3.1)}$$

$$\varepsilon_0 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = I, \quad \varepsilon_1 = \begin{bmatrix} 0 & i \\ -i & 0 \end{bmatrix},$$

$$\varepsilon_2 = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}, \quad \varepsilon_3 = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix},$$

where ε_1, ε_2 and ε_3 are the Pauli spinors of quantum physics. ([49],[50],[51],[52],[53],[54],[55]) The basis elements ε_0, ε_1, ε_2 and ε_3 are the units of the quaternions, which have been used in the spacetime algebra of relativity. The relation between i-type and ε-type hypernumbers are

$$\varepsilon_n^2 = -i_n^2 = I, \quad i_0\varepsilon_n = \varepsilon_n i_0 = i_n,$$

$$i_n\varepsilon_n = \varepsilon_n i_n = i_0, \quad -i_n i_0 = -i_0 i_n = \varepsilon_n,$$

for $n = 1, 2, 3$. ----- (3.3.2)

Consider two matrices A and B, which are separated by a vertical stroke operator (|). The *bimatrix* A|B is defined, consisting of the first domain A (left of the stroke) and the stroke domain B (right of the stroke). The rules for the bimatrices with the i-type and ε-type hypernumbers as the first and the stroke domains are as follows :

$$i_n|\varepsilon = \varepsilon_n|i ,$$

$$i_n|i = -\varepsilon_n|\varepsilon . \qquad \text{----- (3.3.3)}$$

Thus $|i = I|i = \varepsilon_o|i = i_o|\varepsilon$.

Applying these rules, we have the following representations of i-type and ε-type hypernumbers :

$$I = I|I = \varepsilon_o = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} ,$$

$$i_1 = i_1|I = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} ,$$

$$i_2 = i_2|I = \begin{bmatrix} i & 0 \\ 0 & -i \end{bmatrix} ,$$

$$i_3 = i_3|I = \begin{bmatrix} 0 & i \\ i & 0 \end{bmatrix} ,$$

----- (3.3.4)

$$i_4 = I|i_1 = |i_1 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \Bigg| \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix},$$

$$i_5 = i_3|i_1 = \begin{bmatrix} 0 & i \\ i & 0 \end{bmatrix} \Bigg| \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix},$$

$$i_6 = -i_2|i_1 = \begin{bmatrix} -i & 0 \\ 0 & i \end{bmatrix} \Bigg| \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix},$$

$$i_7 = i_1|i_1 = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} \Bigg| \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix},$$

where the last four hypernumbers are expressed as bimatrices, and

$$I_0 = I|i_0 = i_0|I = \begin{bmatrix} i & 0 \\ 0 & i \end{bmatrix},$$

$$\varepsilon_1 = \varepsilon_1|I = \begin{bmatrix} 0 & i \\ -i & 0 \end{bmatrix},$$

$$\varepsilon_2 = \varepsilon_2|I = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix},$$

$$\varepsilon_3 = \varepsilon_3|I = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix},$$

----- (3.3.5)

$$\varepsilon_4 = I|\varepsilon_1 = |\varepsilon = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \Bigg| \begin{bmatrix} 0 & i \\ -i & 0 \end{bmatrix},$$

$$\varepsilon_5 = i_1|\varepsilon_1 = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} \Bigg| \begin{bmatrix} 0 & i \\ -i & 0 \end{bmatrix},$$

$$\varepsilon_6 = -i_2|\varepsilon_1 = \begin{bmatrix} -i & 0 \\ 0 & i \end{bmatrix} \Bigg| \begin{bmatrix} 0 & i \\ -i & 0 \end{bmatrix},$$

$$\varepsilon_7 = i_3|\varepsilon_1 = \begin{bmatrix} 0 & i \\ i & 0 \end{bmatrix} \Bigg| \begin{bmatrix} 0 & i \\ -i & 0 \end{bmatrix}.$$

In the same manner, ε_4, ε_5, ε_6 and ε_7 are bimatrices.

The multiplication of two bimatrices, however, can be explicitly given. Such multiplication is in general *anticommutative* and *antiassociative*. From $i_o|\varepsilon = |i_o\varepsilon$ (thus, the stroke is permeable to i_o and hence *bimatrix-commutative*), we have the following rules for bimatrix multiplication, where A, B, C and D are matrices and exclusion for $\pm\ i_o$ and $\pm\ I$.

$$A|I = A| = A,$$

$$I|B = |B,$$

$$I(\pm I) = \pm I| = \pm I,$$

$$I|I = I^2| = I,$$

$$|i_o = i_o| = i_o,$$

$$A(|B) = -(|B)A = -A|B,$$

$$|B|C = (|B)(|C) = |(BC) = |BC,$$

$$(A|B)^2 = -A^2|B^2 \quad \text{(by anticommutation)},$$

$$(A|A)^2 = -A^2|A^2 \quad \text{(also by anticommutation)},$$

$(i_0|B)^2 = -|B^2$,

$(A|B)|(A|D) = -A^2|BD$,

$(C|B)(A|B) = AC|B^2$,

$A(C|D) = -AC|D = -(C|D)A = CA|D$,

$B(A|A) = (BA|A) = (A|A)B$,

*a $(A|B)(C|D) = AC|BD$,

*b $(C|D)(A|B) = CA|DB$,

(note anticommutation) ,

$(C|D)A = -A(C|D) = AC|D$,

$(A|B)(|B) = -A|B^2 = -(|B)(A|B)$,

$A(A|B) = -(A|B)A = -A^2|B$.

*a,*b Two separate anticommutations yield a net positive result .

We can show that the multiplication of bimatrices is, in general, antiassociative, by applying the above rules to the hypernumbers, such as the units of the octonions, that is,

$(i_1 i_3)(i_1|i) = -(i_2)(i_1|i) = -i_3|i = -i_7$,

$i_1(i_3(i_1|i)) = i_1(-i_2|i) = i_3|i = +i_7$,

$i_5 i_6 = i_3$ or $(i_1|i)(-i_2|i) = -i_3|-i = i_3$.

Bimatrices can be used also in the set of ε-type union i-type hypernumbers, which has 16-space arithmetic.

Table 3.3 The multiplication of the bimatrices with the octonionic units as domains

	$I\vert I$ I	$i_1\vert I$ i_1	$i_2\vert I$ i_2	$i_3\vert I$ i_3	$I\vert i_1$ i_4	$i_3\vert i_1$ i_5	$(-i_2\vert i_1)$ i_6	$i_1\vert i_1$ i_7
$I\vert I$ I	$I\vert I$ I	$i_1\vert I$ i_1	$i_2\vert I$ i_2	$i_3\vert I$ i_3	$I\vert i_1$ i_4	$i_3\vert i_1$ i_5	$(-i_2\vert i_1)$ i_6	$i_1\vert i_1$ i_7
$i_1\vert I$ i_1	$i_1\vert I$ i_1	$i_1^2\vert I$ $-(I\vert I)$ $-I$	$i_1 i_2\vert I$ $i_3\vert I$ i_3	$i_1 i_3\vert I$ $-(i_2\vert I)$ $-i_2$	$i_1\vert i_1$ i_7	$-i_1 i_3\vert i_1$ $i_2\vert i_1$ $-i_6$	$i_1 i_2\vert i_1$ $i_3\vert i_1$ i_5	$i_1^2\vert i_1$ $-(I\vert i_1)$ $-i_4$
$i_2\vert I$ i_2	$i_2\vert I$ i_2	$i_2 i_1\vert I$ $-(i_3\vert I)$ $-i_3$	$i_2^2\vert I^2$ $-(I\vert I)$ $-I$	$i_2 i_3\vert I$ $i_1\vert I$ i_1	$-(i_2\vert i_1)$ i_6	$i_2 i_3\vert i_1$ $i_1\vert i_1$ i_7	$i_2^2\vert i_1$ $-(I\vert i_1)$ $-i_4$	$i_2 i_1\vert i_1$ $-(i_3\vert i_1)$ $-i_5$
$i_3\vert I$ i_3	$i_3\vert I$ i_3	$i_3 i_1\vert I$ $i_2\vert I$ i_2	$i_3 i_2\vert I$ $-(i_1\vert I)$ $-i_1$	$i_3^2\vert I^2$ $-(I\vert I)$ $-I$	$-(i_3\vert i_1)$ $-i_5$	$-i_3^2\vert i_1$ $I\vert i_1$ i_4	$-i_3 i_2\vert i_1$ $i_1\vert i_1$ i_7	$i_3 i_1\vert i_1$ $i_2\vert i_1$ $-i_6$
$I\vert i_1$ i_4	$I\vert i_1$ i_4	$-(i_1\vert i_1)$ $-i_7$	$i_2\vert i_1$ $-i_6$	$i_3\vert i_1$ i_5	$I^2\vert i_1^2$ $-(I\vert I)$ $-I$	$i_3\vert i_1^2$ $-(i_3\vert I)$ $-i_3$	$-i_2\vert i_1^2$ $i_2\vert I$ i_2	$-i_1\vert i_1^2$ $i_1\vert I$ i_1
$i_3\vert i_1$ i_5	$i_3\vert i_1$ i_5	$-i_3 i_1\vert i_1$ $-(i_2\vert i_1)$ i_6	$i_3 i_2\vert i_1$ $-i_1\vert i_1$ $-i_7$	$i_3^2\vert i_1$ $-(I\vert i_1)$ $-i_4$	$-i_3\vert i_1^2$ $i_3\vert I$ i_3	$-i_3^2\vert i_1^2$ $-(I\vert I)$ $-I$	$-i_3 i_2\vert i_1^2$ $-(i_1\vert I)$ $-i_1$	$-i_3 i_1\vert i_1^2$ $i_2\vert I$ i_2
$-i_2\vert i_1$ i_6	$-i_2\vert i_1$ i_6	$i_2 i_1\vert i_1$ $-(i_3\vert i_1)$ $-i_5$	$-i_2^2\vert i_1$ $I\vert i_1$ i_4	$-i_2 i_3\vert i_1$ $-(i_1\vert i_1)$ $-i_7$	$i_2\vert i_1^2$ $-(i_2\vert I)$ $-i_2$	$-i_2 i_3\vert i_1^2$ $i_1\vert I$ i_1	$-i_2^2\vert i_1^2$ $-(I\vert I)$ $-I$	$-i_2 i_1\vert i_1^2$ $i_3\vert I$ i_3
$i_1\vert i_1$ i_7	$i_1\vert i_1$ i_7	$-i_1^2\vert i_1$ $I\vert i_1$ i_4	$i_1 i_2\vert i_1$ $i_3\vert i_1$ i_5	$i_1 i_3\vert i_1$ $-(i_2\vert i_1)$ i_6	$i_1\vert i_1^2$ $-(i_1\vert I)$ $-i_1$	$-i_1 i_3\vert i_1^2$ $-i_2\vert I$ $-i_2$	$i_1 i_2\vert i_1^2$ $-(i_3\vert I)$ $-i_3$	$-i_1^2\vert i_1^2$ $-(I\vert I)$ $-I$

Section 3.4 Power-Associative Products on Octonions

Myung has studied Malcev-admissible algebras and an associative product on octonions, and considered the property of the operator :

$$A * B = \frac{1}{2}[A,B] + \tau(A)B + \tau(B)A + \sigma(A,B)e \quad , \qquad \text{------ (3.4.1)}$$

$A,B \in M$, $M = \begin{bmatrix} a & X \\ Y & b \end{bmatrix}$, where $\tau(M)$ is a linear function and σ is a bilinear form. ([4],[24],[43],[44],[45],[57],[58]) We will study the cases of $\tau(M)$ and $\sigma(A,B)$ that make the product $A*B$ power associative or flexible, by giving examples of $\tau(M)$ such as

$$\begin{aligned} \tau(M) &= 0 \quad , \\ \tau(M) &= ab \quad , \qquad \text{----- (3.4.2)} \\ \tau(M) &= |X||Y| \quad , \\ \tau(M) &= \sqrt{a^2 + b^2 + |X|^2 + |Y|^2} \quad , \end{aligned}$$

and compute the results by using a computer.

3.4.1 Malcev-admissible algebras

DEFINITION : An algebra M with multiplication [A,B] over a field F of arbitrary characteristic is called a *Malcev algebra* if it satisfies the anticommutative law [A,B] = - [B,A] (which implies [A,A] = 0), and the Malcev identity

$$[[A,B],[A,C]] = [[[A,B],C],A]+[[[B,C],A],A]+[[[C,A],A],B] , \quad \text{----- (3.4.3)}$$

for all $A,B,C \in M$. An algebra L over F is said to be *Malcev admissible* if the attached minus algebra L^- is a Malcev algebra.

Let L denote the associative algebra of $n \times n$ matrices over a field F of characteristic not 2 or 3, and let AB denote the matrix product in A. We write $(L,*)$ for the algebra with multiplication $A * B$ defined on the same vector space as L, and $(L,*)$ for all the algebras that satisfy the third power identity

$$(A * A) * A = A * (A * A) , \quad \text{----- (3.4.4)}$$

and the relation

$$[A,B]^* = A * B - B * A = [A,B] = AB - BA , \quad \text{----- (3.4.5)}$$

for all $A,B \in L$. The multiplication $A * B$ in such an algebra $(L,*)$ is described by

$$A * B = \frac{1}{2}[A,B] + \tau(A)B + \tau(B)A + \beta(AB + BA) + \sigma(A,B)e , \quad \text{----- (3.4.6)}$$

for a linear functional $\tau : L \longrightarrow F$ and a symmetric bilinear form $\sigma : L \times L \longrightarrow F$, where e is the identity element of L. In the case $n = 2$, $\beta \in F$ can be taken to be zero. $(L,*)$ is thus a power-associative product which depends on the relation between τ and σ.

3.4.2 Third and fourth power-associative products

Let $\mathbf{O} = \mathbf{O}(\alpha,\beta,\gamma)$ be an octonion algebra over a field F of characteristic $\neq 2$. Let t be the trace in $\mathbf{O}$, and let $V = \{ A \in \mathbf{O} \mid t(A) = 0 \}$. Then, $\mathbf{O} = F_e \oplus V$, and V is closed under the product [,], where e is the identity element of $\mathbf{O}$. It is well known that V^- is a 7-dimensional central simple Malcev algebra for $\alpha\beta\gamma \neq 0$. We can choose a basis $e_1, e_2, \ldots, e_7$ of V so that V^- has the following multiplication table (see Table 3.4A).

Table 3.4A The multiplication table of the 7-dimensional central simple Malcev algebra

	e_1	e_2	e_3	e_4	e_5	e_6	e_7
e_1	$-\alpha e$	$2e_3$	$-2\alpha e_2$	$2e_5$	$-2\alpha e_4$	$-2e_7$	$2\alpha e_6$
e_2		$-\beta e$	$2\beta e_1$	$2e_6$	$2e_7$	$-2\beta e_4$	$-2\beta e_5$
e_3			$-\alpha\beta\gamma$	$2e_7$	$-2\alpha e_6$	$2\beta e_5$	$-2\alpha\beta e_4$
e_4				$-\gamma e$	$2\gamma e_1$	$2\gamma e_2$	$2\gamma e_3$
e_5					$-\alpha\gamma e$	$-2\gamma e_3$	$2\alpha\gamma e_2$
e_6						$-\beta\gamma e$	$-2\beta\gamma e_1$
e_7							$-\alpha\beta\gamma e$

The off-diagonal entries represent the products in V^- and the diagonal entries are all non-vanishing Jordan products

$$A \cdot B = \frac{1}{2}(AB + BA) , \qquad \text{-----} \ (3.4.7)$$

for $A, B = e_1, \ldots, e_7$.

Since $$AB = \frac{1}{2}[A,B] + A \cdot B , \qquad \text{-----} \ (3.4.8)$$

the products in $\mathbf{O}$ are readily obtained in Table 3.4A.

If $A * B$ is a third power-associative product defined on the vector space V over F of characteristic $\neq 2$, with

$$[A,B]^* = A * B - B * A = [A,B] = AB - BA ,$$

then the product $A * B$ is given by

$$A * B = \frac{1}{2}[A,B] + \tau(A)B + \tau(B)A , \qquad \text{-----} \ (3.4.9)$$

for some linear functional τ . A linear functional τ on V is third power-associative.

If $\mathbf{O}$ is an octonion algebra over a field F of characteristic $\neq 2$ and $A * B$ is a third power-associative product defined on $\mathbf{O}$ such that

$$[A,B]^* = A * B - B * A = -[A,B] , \qquad \text{-----} \ (3.4.10)$$

then there exist a linear functional τ on $\mathbf{O}$ and a symmetric bilinear form σ on $\mathbf{O}$ such that

$$A * B = \frac{1}{2}[A,B] + \tau(A)B + \tau(B)A + \sigma(A,B)e \text{ ,-----} \quad (3.4.11)$$

for all $A,B \in L$, where e is the identity element of $\mathbf{O}$. Conversely, any product given by Eq.(3.4.11), with a linear functional τ on $\mathbf{O}$ and a symmetric bilinear form σ on $\mathbf{O}$, is third power-associative. But $(\mathbf{O},*)$ is fourth power-associative if and only if one of the following conditions holds :

(a) τ is arbitrary and $\sigma = 0$;

(b) σ is arbitrary and τ is given by $\tau(A) = -\sigma(e,A)$;

(c) τ and σ satisfy the relation $\tau(A)\tau(B) - \sigma(e,A)\sigma(e,B) + \sigma(A,B)\sigma(e,e) = 0$.

Now $(\mathbf{O},\cdot) = (\mathbf{O},*)^{+}$ is a quadratic Jordan algebra. Thus, $(\mathbf{O},*)$ is *Jordan-admissible* as well as *Malcev-admissible*. If the characteristic of F is not 2 or 3, then $(\mathbf{O},*)$ is power-associative if and only if one of the conditions (a) - (c) holds.

3.4.3 Third and fourth power-associative products on split octonions

Recall the split octonions considered in Section 1. Let A be a vector matrix

$$A = \begin{bmatrix} a & X \\ Y & b \end{bmatrix}, \quad a,b \in \mathbb{R} \quad \text{and} \quad X,Y \in \mathbb{R}^3,$$

where $\mathbb{R}^3$ is three-dimensional vector space. If the basis of A is given as follows :

$$U_0 = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \quad U_0^* = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix},$$

$$U_i = \begin{bmatrix} 0 & e_i \\ 0 & 0 \end{bmatrix}, \quad U_i^* = \begin{bmatrix} 0 & 0 \\ e_i & 0 \end{bmatrix}, \quad \text{------ (3.4.12)}$$

then A is an algebra of split octonions.

Consider Eq.(3.4.6) :

$$A * B = \frac{1}{2}[A,B] + \tau(A)B + \tau(B)A + \beta(AB + BA) + \sigma(A,B)e,$$

for $A,B \in L$, $[A,B] = AB - BA$, with a linear functional $\tau : L \longrightarrow F$ and with a symmetric bilinear form $\sigma : L \times L \longrightarrow F$, where e is the identity element of L. In this case β can be taken to be zero since A is a 2 X 2 vector matrix algebra. Then the algebra $(L,*)$ is third power-associative if and only if the product "$*$" is given by

$$A * B = \frac{1}{2}[A,B] + \tau(A)B + \tau(B)A + c(AB), \quad \text{------ (3.4.13)}$$

for all $A,B \in L$, where τ is a linear function on L, and c is a scalar in F. But the algebra

$(L,*)$ is fourth power-associative product if and only if one of the following conditions holds :

(a) $\sigma = 0$ and $\tau(L) = \sqrt{a^2 + b^2 + |X|^2 + |Y|^2}$

$|X|$ and $|Y|$ are magnitudes of X and Y, respectively ;

(b) $\tau = 0$ and σ is arbitrary.

The algebra $(L,*)$ with multiplication defined by Eq.(3.4.6) is *flexible* if and only if the linear functional τ on L is identically zero. Then

$$A * B = \frac{1}{2}[A,B] + c(AB) , \qquad ----- (3.4.14)$$

for all $A,B \in L$, where c is a fixed scalar.

Tables 3.4B - 3.4E are the results of numerical computation.

Table 3.4B

$$A * B = \frac{1}{2}[A,B] + \tau(A)B + \tau(B)A$$

$$[A,B] = AB - BA$$

If $A = \begin{bmatrix} a & X \\ Y & b \end{bmatrix}$, $B = \begin{bmatrix} c & U \\ V & d \end{bmatrix}$,

then $AB = \begin{bmatrix} ac + (X \cdot V) & g(d)X + g(a)U + t(Y \wedge V) \\ g(c)Y + g(b)V + s(X \wedge U) & bd + (Y \cdot U) \end{bmatrix}$,

where s and t are scalars in F, and the linear function $g : F \longrightarrow F$ is given.

$\tau(A)$ / property	0	ab	$\lvert X\rvert \lvert Y\rvert$	$\sqrt{a^2+b^2+\lvert X\rvert^2+\lvert Y\rvert^2}$
flexible	√	x	x	x
antiflexible	x	x	x	x
3^{rd} power-associative	√	√	√	√
4^{th} power-associative	√	x	x	√

Table 3.4C

$$A * B = \frac{1}{2}[A,B] + \tau(A)B + \tau(B)A$$

$$[A,B] = AB - BA$$

If $A = \begin{bmatrix} a & X \\ Y & b \end{bmatrix}$, $B = \begin{bmatrix} c & U \\ V & d \end{bmatrix}$,

then $AB = \begin{bmatrix} ac - (X \cdot V) & dX + aU + (Y \wedge V) \\ cY + bV + (X \wedge U) & bd - (Y \cdot U) \end{bmatrix}$.

property \ $\tau(A)$	0	ab	$\lvert X\rvert\lvert Y\rvert$	$\sqrt{a^2+b^2+\lvert X\rvert^2+\lvert Y\rvert^2}$
flexible	√	x	x	x
antiflexible	x	x	x	x
3rd power-associative	√	√	√	√
4th power-associative	√	x	x	√

Table 3.4D

$$A * B = \frac{1}{2}[A,B] + \tau(A)B + \tau(B)A + c(AB)$$

$[A,B] = AB - BA$ and c is a scalar in $\mathbb{R}$

If $A = \begin{bmatrix} a & \mathbf{X} \\ \mathbf{Y} & b \end{bmatrix}$, $B = \begin{bmatrix} c & \mathbf{U} \\ \mathbf{V} & d \end{bmatrix}$,

then $AB = \begin{bmatrix} ac - (\mathbf{X}\cdot\mathbf{V}) & d\mathbf{X} + a\mathbf{U} + (\mathbf{Y}\wedge\mathbf{V}) \\ c\mathbf{Y} + b\mathbf{V} + (\mathbf{X}\wedge\mathbf{U}) & bd - (\mathbf{Y}\cdot\mathbf{U}) \end{bmatrix}$.

$\tau(A)$ / property	0	ab	$\lvert X\rvert\lvert Y\rvert$	$\sqrt{a^2+b^2+\lvert X\rvert^2+\lvert Y\rvert^2}$
flexible	√	x	x	x
antiflexible	x	x	x	x
3^{rd} power-associative	√	√	√	√
4^{th} power-associative	√	x	x	x

Table 3.4E

$$A * B = \frac{1}{2}[A,B] + \tau(A)B + \tau(B)A + c(AB)$$

$[A,B] = AB - BA$ and c is a scalar in $\mathbb{R}$

If $A = \begin{bmatrix} a & X \\ Y & b \end{bmatrix}$, $B = \begin{bmatrix} c & U \\ V & d \end{bmatrix}$,

then $AB = \begin{bmatrix} ac + (X \cdot V) & g(a)U + g(d)X \\ g(b)V + g(c)Y & bd + (Y \cdot U) \end{bmatrix}$,

where g is the linear function, $g : F \longrightarrow F$.

$\tau(A)$ / property	0	ab	$\lvert X\rvert\lvert Y\rvert$	$\sqrt{a^2+b^2+\lvert X\rvert^2+\lvert Y\rvert^2}$
flexible	√	x	x	x
antiflexible	x	x	x	x
3^{rd} power-associative	√	√	√	√
4^{th} power-associative	√	x	x	x

CHAPTER IV

THE REPRESENTATION OF VECTOR MATRICES WITH THE PICTOGRAPHICAL COMPUTER METHOD

This chapter gives an account of the representation of vector matrices with the pictographical computer method. (See Appendix A)

For simple three-dimensional graphics, the right-handed Cartesian coordinate system (X,Y,Z) may be aligned such that the Y axis is horizontal in the output display rectangle from left to right, the Z axis is vertical running from the bottom of the display rectangle to the top, and the X axis, pointing upward , is thus perpendicular to the display surface. (see Fig. 4A)

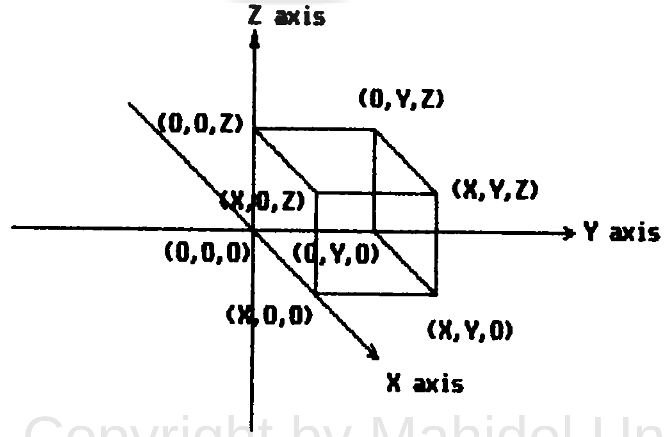


Fig. 4A The right handed Cartesian coordinate system and the display rectangle

A simple and natural way for mapping from the three-dimensional coordinate space onto the display rectangle is given by

$$(X,Y,Z) \longrightarrow (X',Y') \;;$$

that is, the Z' component is ignored and (X',Y') is used in the manner of two-dimensional graphics. ([64])

Consider the form of the vector matrix **M**,

$$\mathbf{M} = \begin{bmatrix} a & \mathbf{U} \\ \mathbf{V} & b \end{bmatrix}, \quad \text{for} \quad a,b \in \mathbb{R}, \quad \mathbf{U},\mathbf{V} \in \mathbb{V},$$

where $\mathbf{U} = U_x\mathbf{i} + U_y\mathbf{j} + U_z\mathbf{k}$ and $\mathbf{V} = V_x\mathbf{i} + V_y\mathbf{j} + V_z\mathbf{k}$. The fixed vectors **U** and **V** in three-dimensional space both start from the origin (0,0,0), and go to point (U_x,U_y,U_z) and point (V_x,V_y,V_z), respectively.

The representation of the vector matrix **M** is shown in Fig. 4B below.

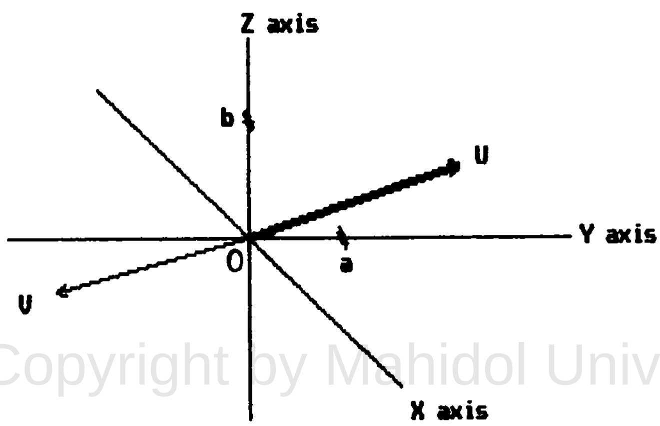


Fig. 4B The representation of a vector matrix **M** in three-dimensional coordinate space

The values of a and b are represented by asterisks on Y and Z axes, respectively. The vectors U and V are represented by bold line and ordinary line, respectively. The projections of the vectors U and V in the XY, YZ, and XZ planes are represented by dash lines as shown in Figs. 4C - 4D. The three projective lines for each vector are mutually orthogonal.

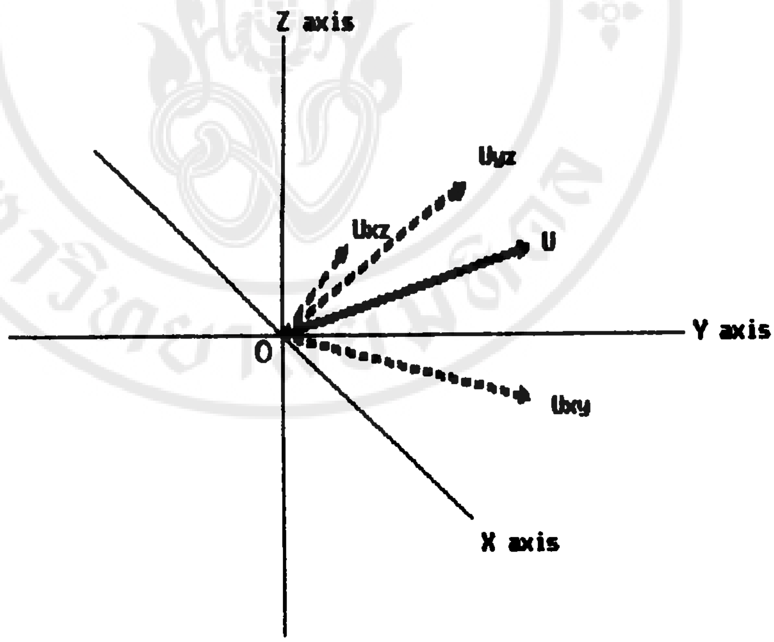


Fig. 4C The projection of U in three-dimensional coordinate space

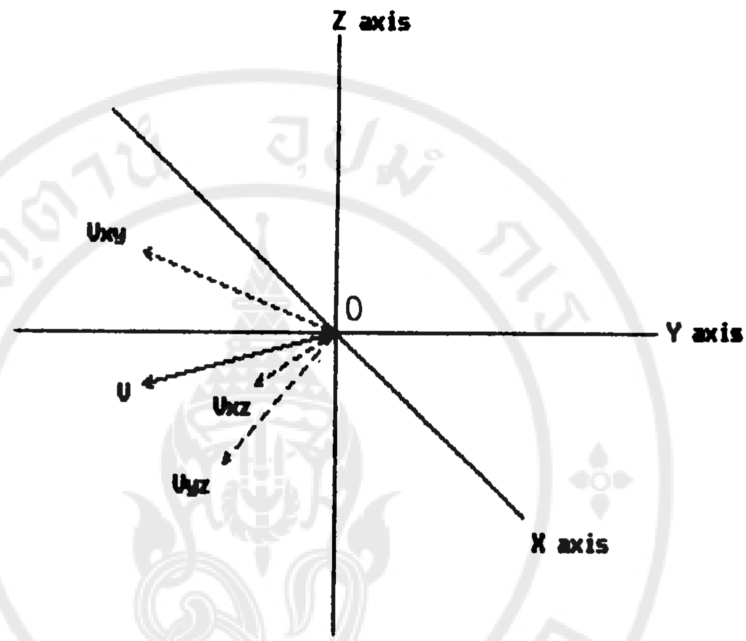


Fig. 4D The projection of V in three-dimensional coordinate space

The connecting lines of U and its projective lines (U_{xy} , U_{yz} , and U_{xz}) , and the projective lines on the X , Y , and Z axes, are drawn in such a way that the components of the vector U are clearly seen (see Fig. 4E). Similarly, the projections of the vector V on all the axes are shown in Fig. 4F.

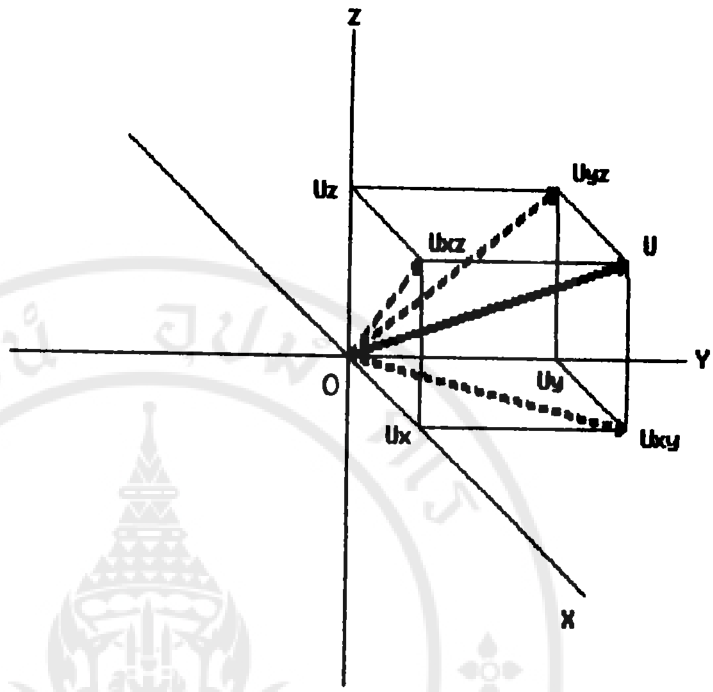


Fig. 4E The projections of vector U on all axes

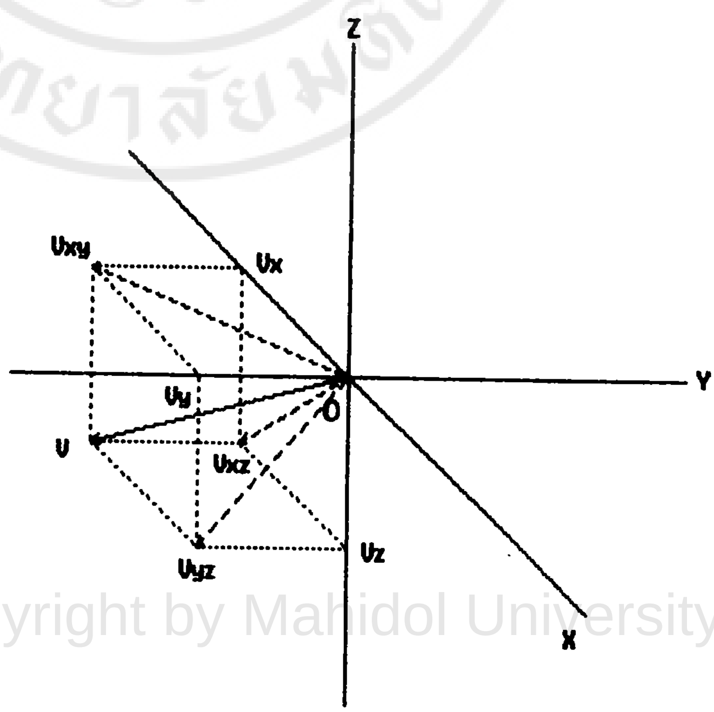


Fig. 4F The projections of vector V on all axes

The construction of perspectives of planes and octants for graphical display is illustrated in Fig. 4G

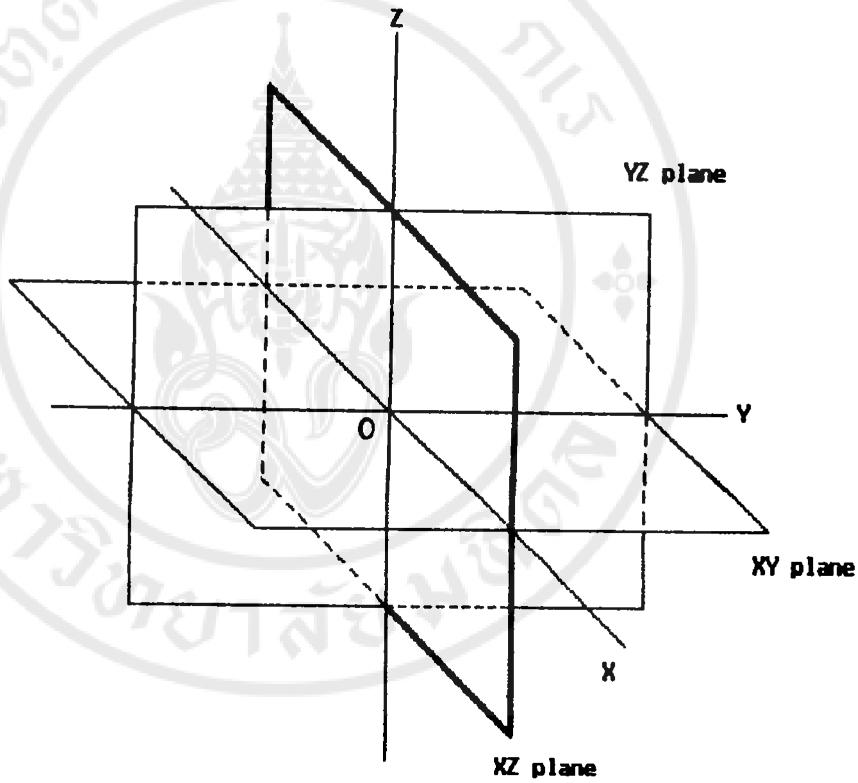


Fig. 4G Three mutually orthogonal planes

In Fig. 4H is shown the combination of the seven preceding figures (Figs. 4A - 4G).

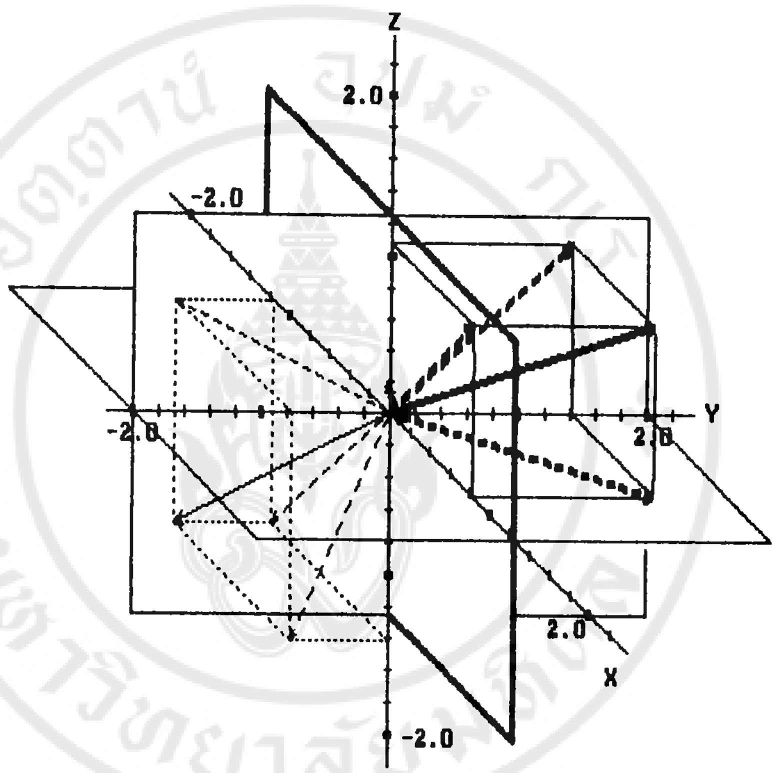


Fig. 4H The combination of the seven preceding figures

In the following, the representations of the vector matrices $\begin{bmatrix} a & X \\ Y & b \end{bmatrix}$, and $\begin{bmatrix} c & U \\ V & d \end{bmatrix}$, are shown in Figs. (a) and (b), respectively.

The Zorn product $C = A *_z B$

where $$C = \begin{bmatrix} ac - X \cdot V & aU + dX + Y \wedge V \\ cY + bV + X \wedge U & bd + Y \cdot V \end{bmatrix},$$

the Fellouris product $D = A *_F B$

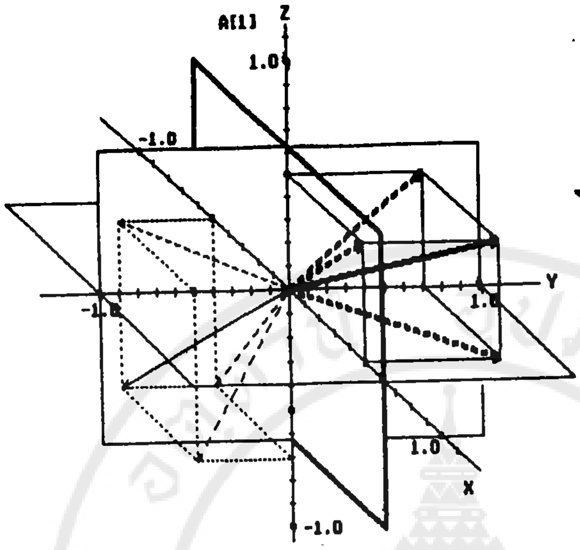
where $$D = \begin{bmatrix} ac - X \cdot V & aU + dX \\ cY + bV & bd - Y \cdot V \end{bmatrix},$$

and the Myung product

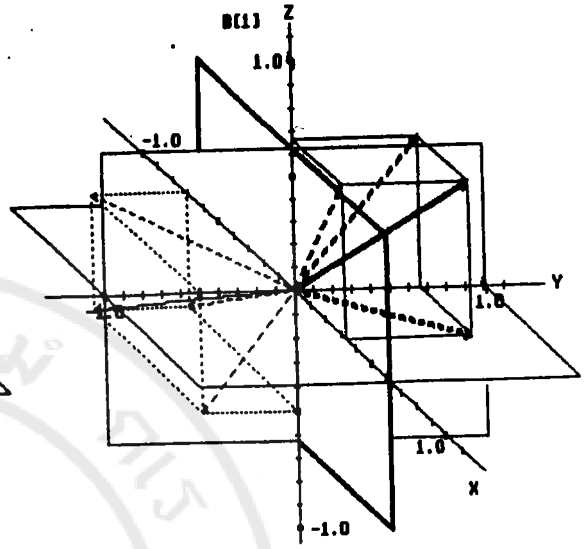
$$A *_M B = \tfrac{1}{2}[A, B] + \tau(A)B + \tau(B)A$$

where $[A, B] = AB - BA$, and AB and BA are, respectively, the Zorn or the Fellouris product, are then presented. The choice of the linear functional τ is considered in chapter III. The pictographical representations of the Myung-Zorn products, E (with $\tau = 0$) and F (with

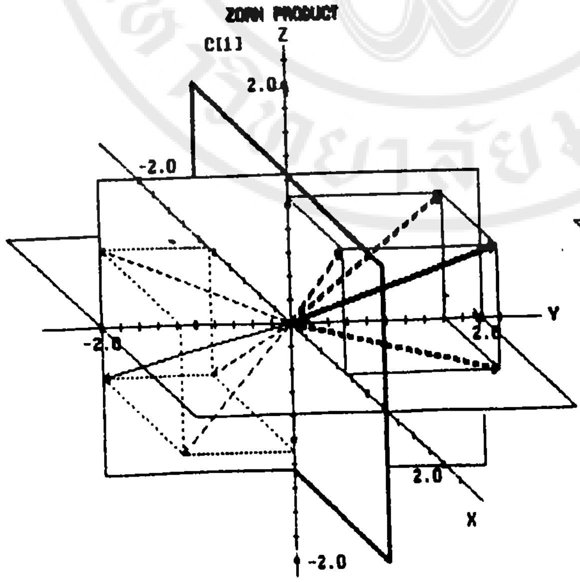
$\tau = \sqrt{a^2 + b^2 + |X|^2 + |Y|^2}$), are displayed.


A[1]
Z
1.0
-1.0
Y
-1.0
1.0
1.0
X
-1.0

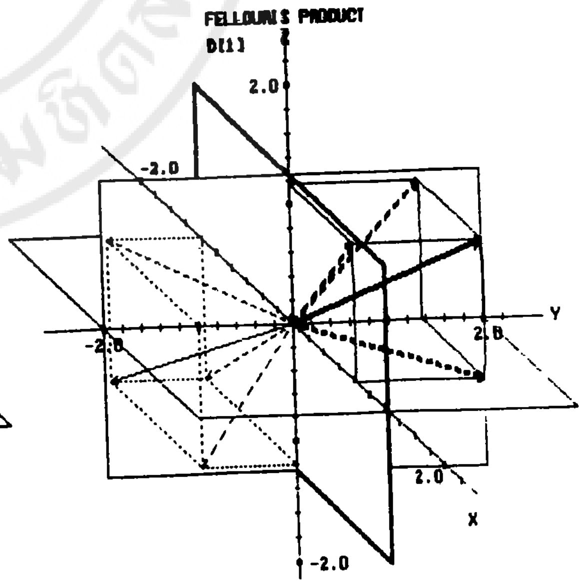
(a)


B[1]
Z
1.0
-1.0
Y
1.0
1.0
X
-1.0

(b)


ZORN PRODUCT
C[1]
Z
2.0
-2.0
Y
-2.0
2.0
2.0
X
-2.0

(c)


D[1]
Z
2.0
-2.0
Y
-2.0
2.0
2.0
X
-2.0

(d)

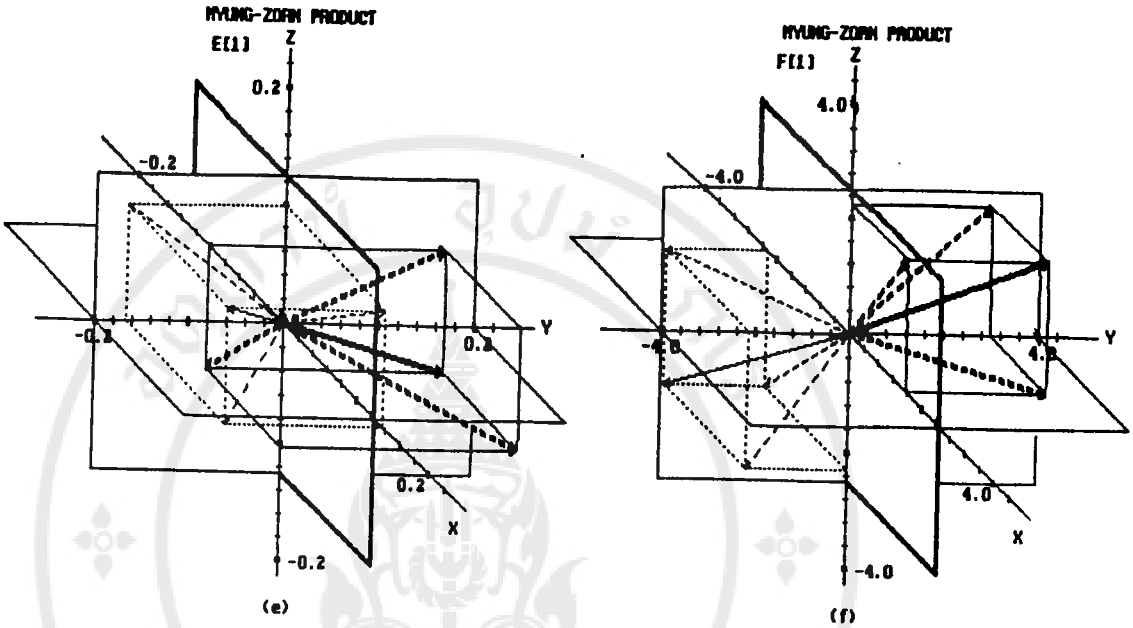
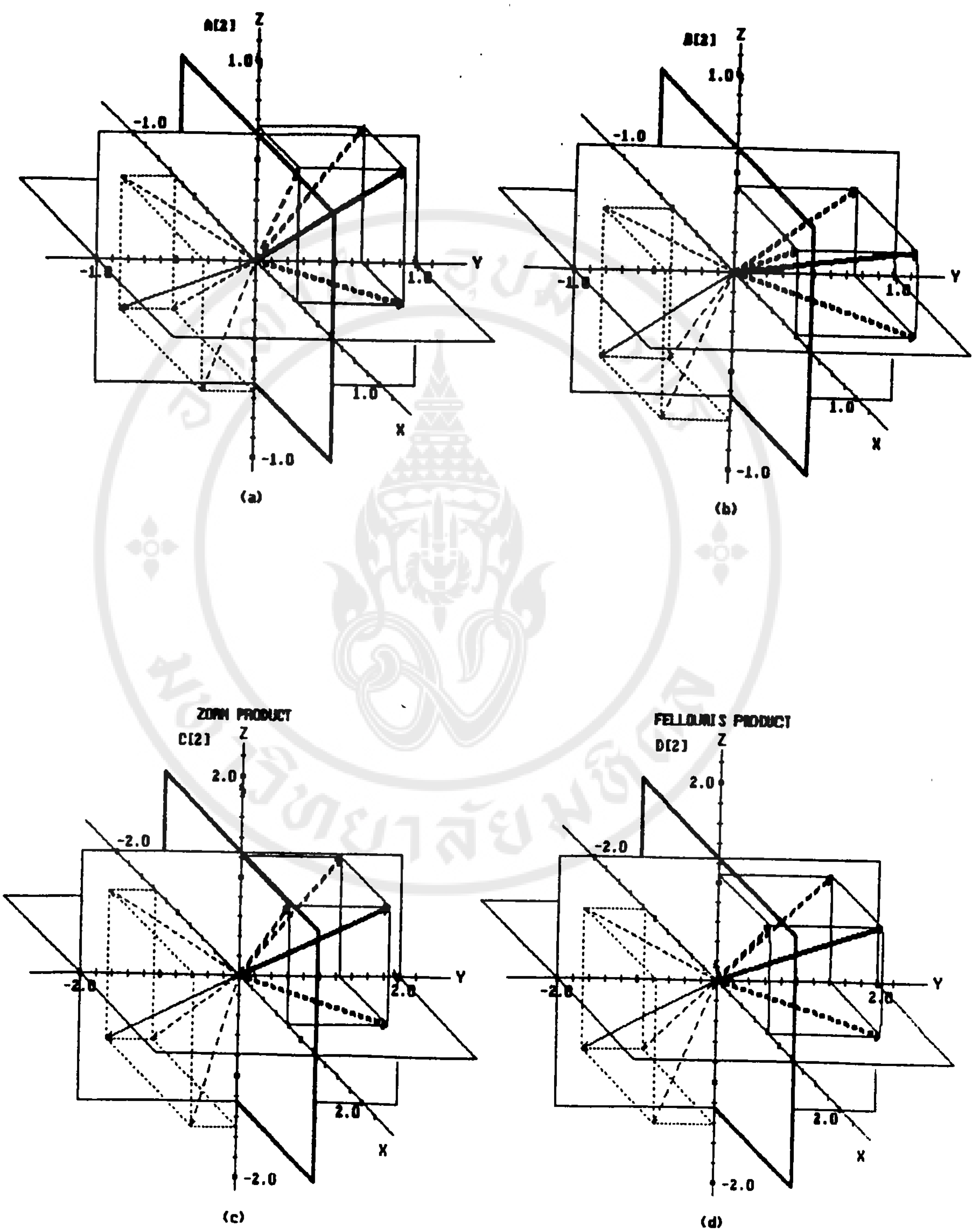


Fig. 4I(1) The representations of the vector matrices A , B , ZORN PRODUCT C , FELLOURIS PRODUCT D , MYUNG-ZORN PRODUCT E ($\tau = 0$) , and MYUNG-ZORN PRODUCT F ($\tau = \sqrt{a^2 + b^2 + |U^2| + |V^2|}$) , where A , B , C , D , E , and F $\in$ M (M = $\begin{bmatrix} a & U \\ V & b \end{bmatrix}$) , for data set 1. The scalars a and b are represented by asterisks on the Y and Z axes. The vectors U and V are represented by bold lines and ordinary lines, respectively.


A[2]
Z
1.0
-1.0
-1.0
1.0
Y
1.0
X
-1.0
(a)
B[2]
Z
1.0
-1.0
-1.0
1.0
Y
1.0
X
-1.0
(b)
ZORN PRODUCT
C[2]
Z
2.0
-2.0
-2.0
2.0
Y
2.0
X
-2.0
(c)
FELLOURIS PRODUCT
D[2]
Z
2.0
-2.0
-2.0
2.0
Y
2.0
X
-2.0
(d)

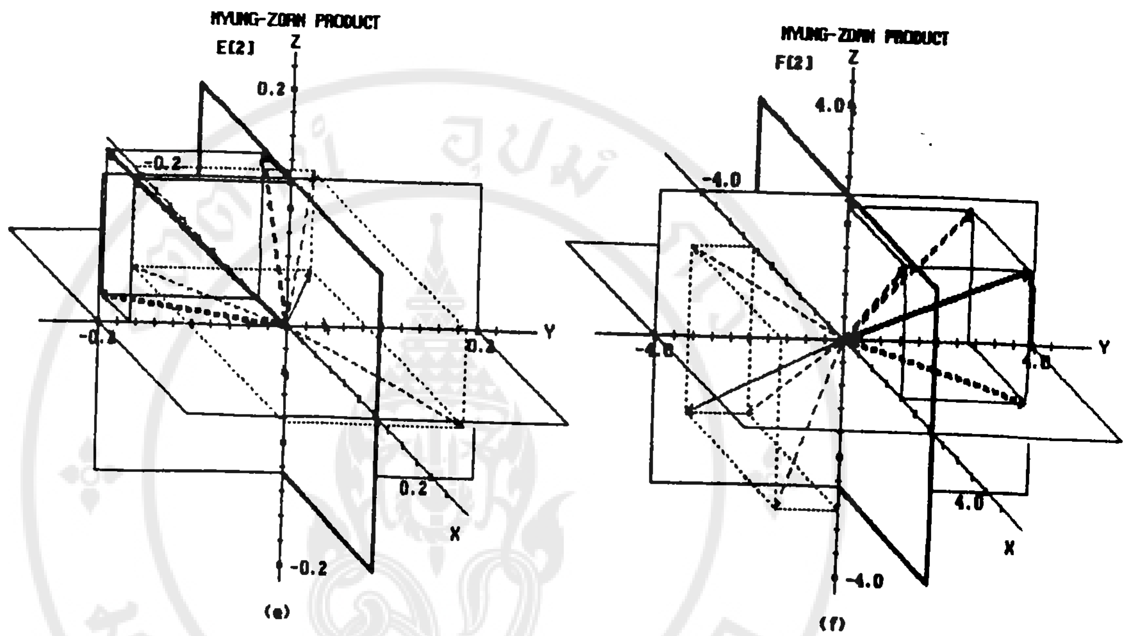
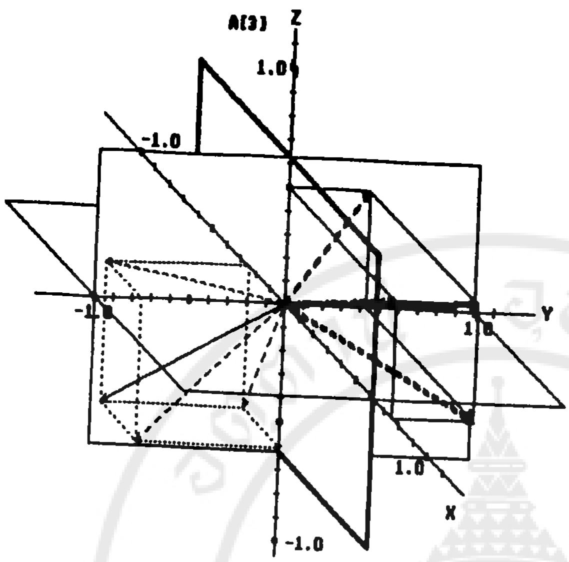
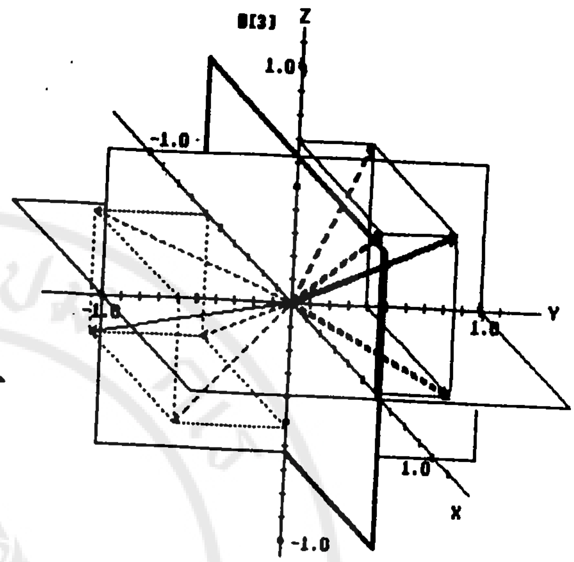


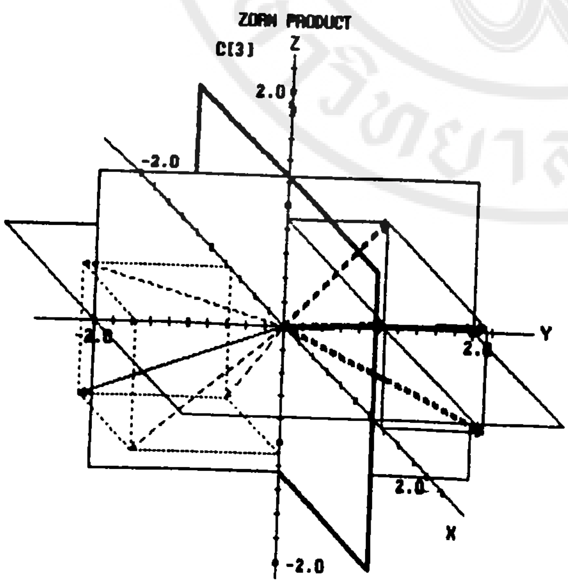
Fig. 41(2) The representations of the vector matrices A , B , ZORN PRODUCT C , FELLOURIS PRODUCT D , MYUNG-ZORN PRODUCT E ($\tau = 0$) , and MYUNG-ZORN PRODUCT F ($\tau = \sqrt{a^2 + b^2 + |U^2| + |V^2|}$) , where A , B , C , D , E , and F $\in$ M ($M = \begin{bmatrix} a & U \\ V & b \end{bmatrix}$) , for data set 2. The scalars a and b are represented by asterisks on the Y and Z axes. The vectors U and V are represented by bold lines and ordinary lines, respectively.


A[3]
Z
1.0
-1.0
Y
X

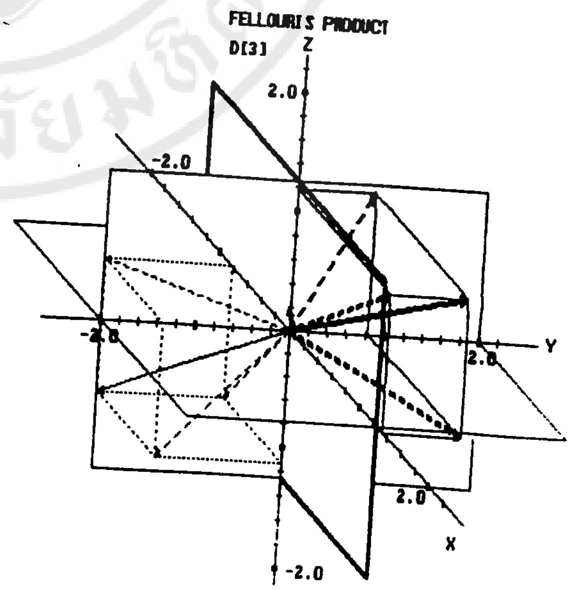
(a)


B[3]
Z
1.0
-1.0
Y
X

(b)


ZORN PRODUCT
C[3]
Z
2.0
-2.0
Y
X

(c)


FELLOURIS PRODUCT
D[3]
Z
2.0
-2.0
Y
X

(d)

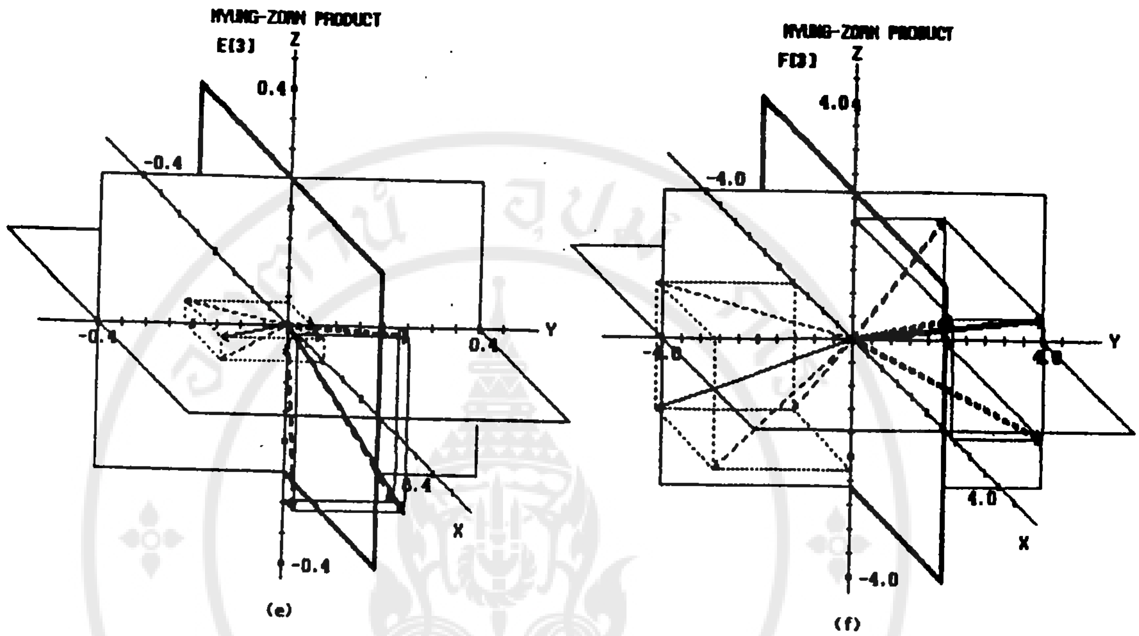
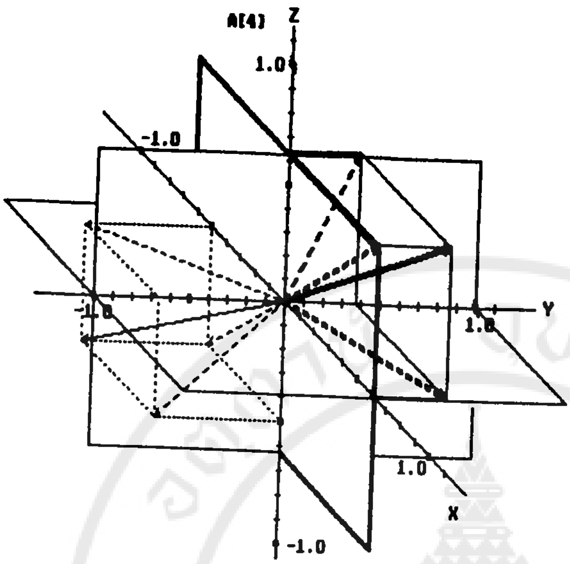
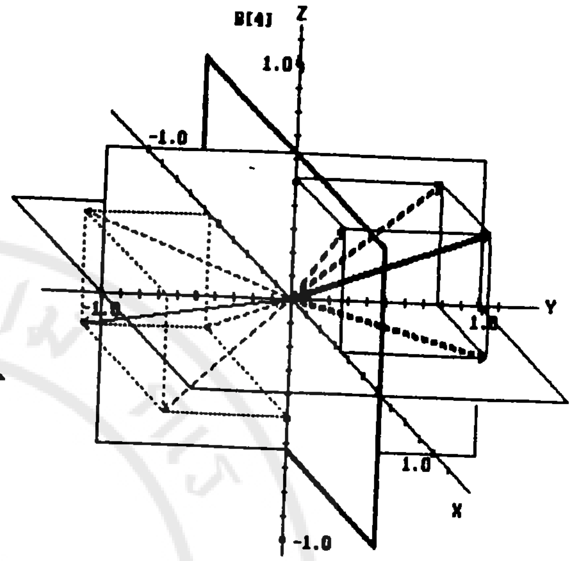


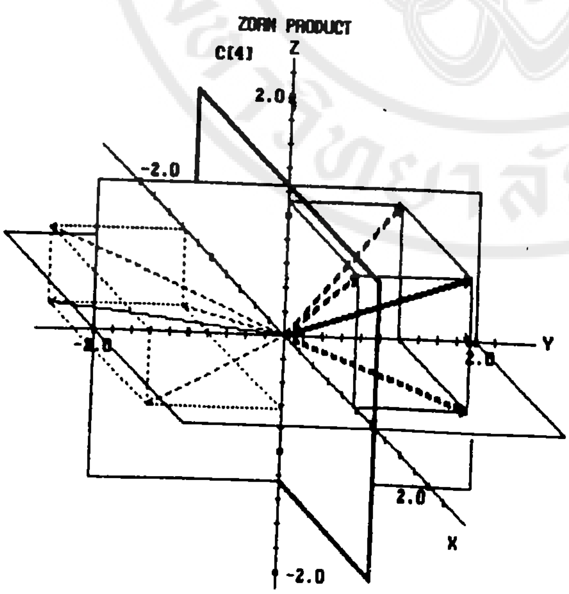
Fig. 4I(3) The representations of the vector matrices A , B , ZORN PRODUCT C , FELLOURIS PRODUCT D , MYUNG-ZORN PRODUCT E ($\tau = 0$) , and MYUNG-ZORN PRODUCT F ($\tau = \sqrt{a^2 + b^2 + |U^2| + |V^2|}$) , where A , B , C , D , E , and F $\in$ M ($M = \begin{bmatrix} a & U \\ V & b \end{bmatrix}$) , for data set 3. The scalars a and b are represented by asterisks on the Y and Z axes. The vectors U and V are represented by bold lines and ordinary lines, respectively.


A[4]
Z
1.0
-1.0
-1.0
1.0
Y
1.0
X
-1.0

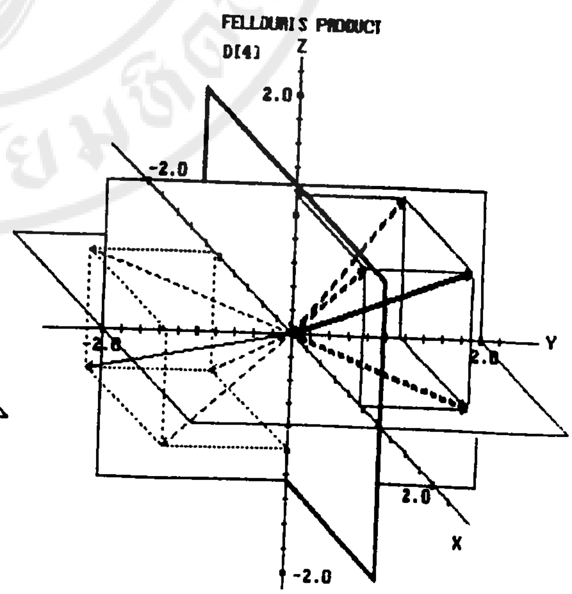
(a)


B[4]
Z
1.0
-1.0
Y
1.0
X
-1.0

(b)


ZORN PRODUCT
C[4]
Z
2.0
-2.0
Y
2.0
X
-2.0

(c)


FELLOURIS PRODUCT
D[4]
Z
2.0
-2.0
Y
2.0
X
-2.0

(d)

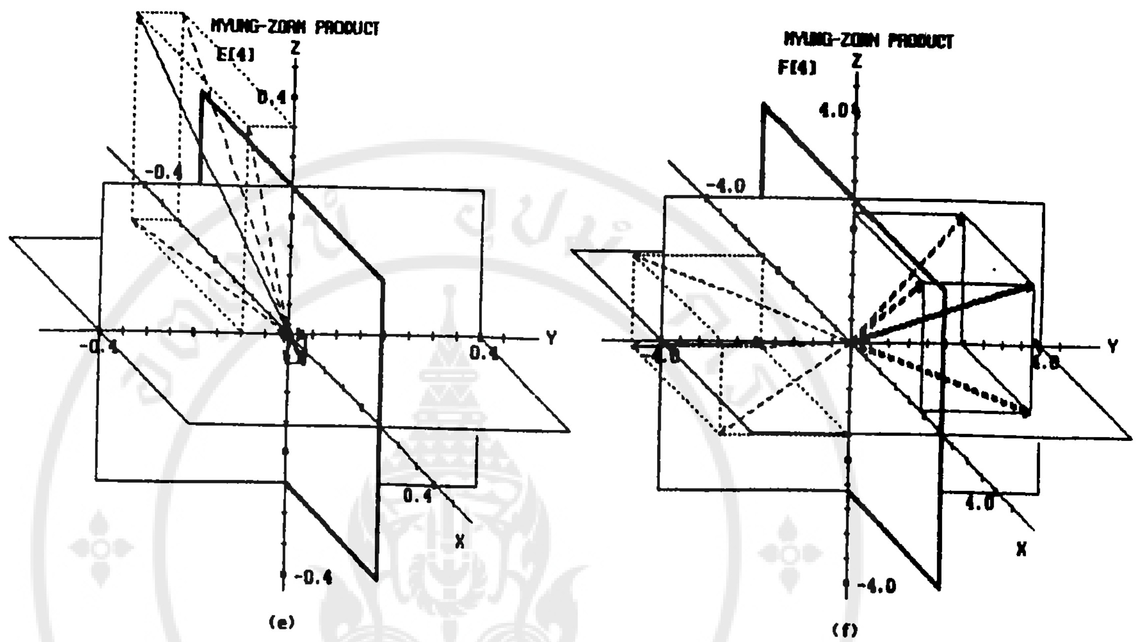
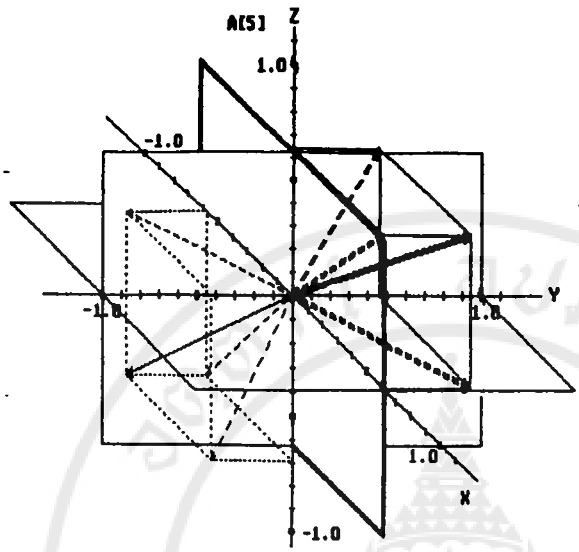
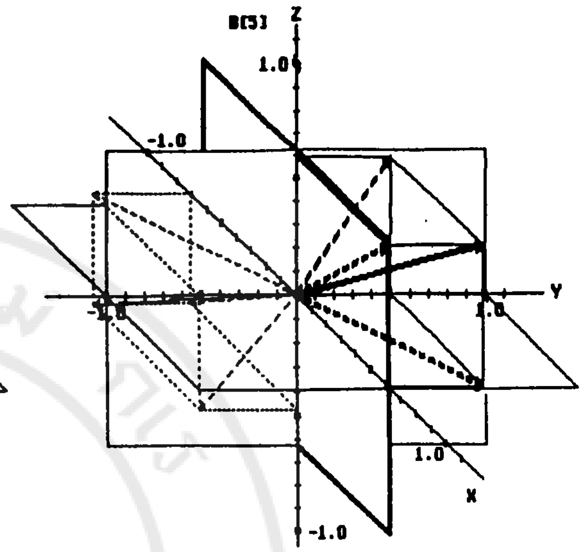


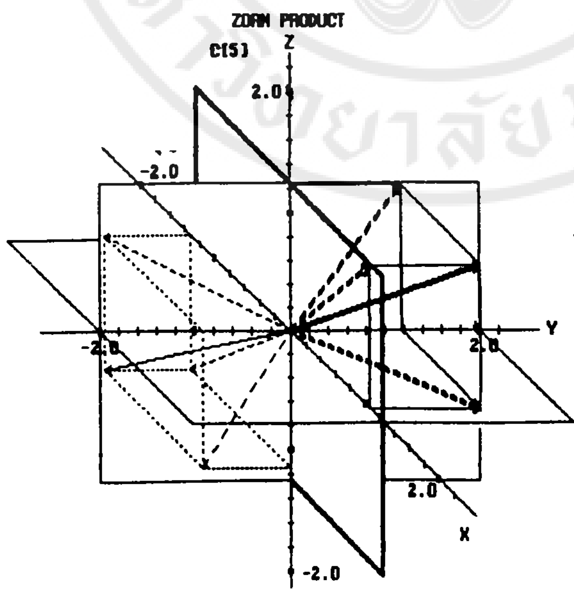
Fig. 4I(4) The representations of the vector matrices **A**, **B**, ZORN PRODUCT **C**, FELLOURIS PRODUCT **D**, MYUNG-ZORN PRODUCT **E** ($\tau = 0$), and MYUNG-ZORN PRODUCT **F** ($\tau = \sqrt{a^2 + b^2 + |U^2| + |V^2|}$), where **A**, **B**, **C**, **D**, **E**, and **F** $\in$ **M** (**M** = $\begin{bmatrix} a & U \\ V & b \end{bmatrix}$), for data set 4. The scalars a and b are represented by asterisks on the Y and Z axes. The vectors U and V are represented by bold lines and ordinary lines, respectively.


A[5]
Z
1.0
-1.0
Y
-1.0
1.0
1.0
X
-1.0

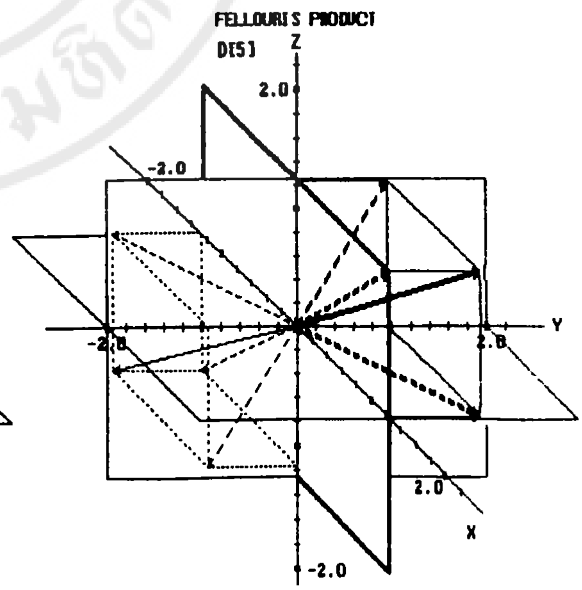
(a)


B[5]
Z
1.0
-1.0
Y
1.0
1.0
X
-1.0

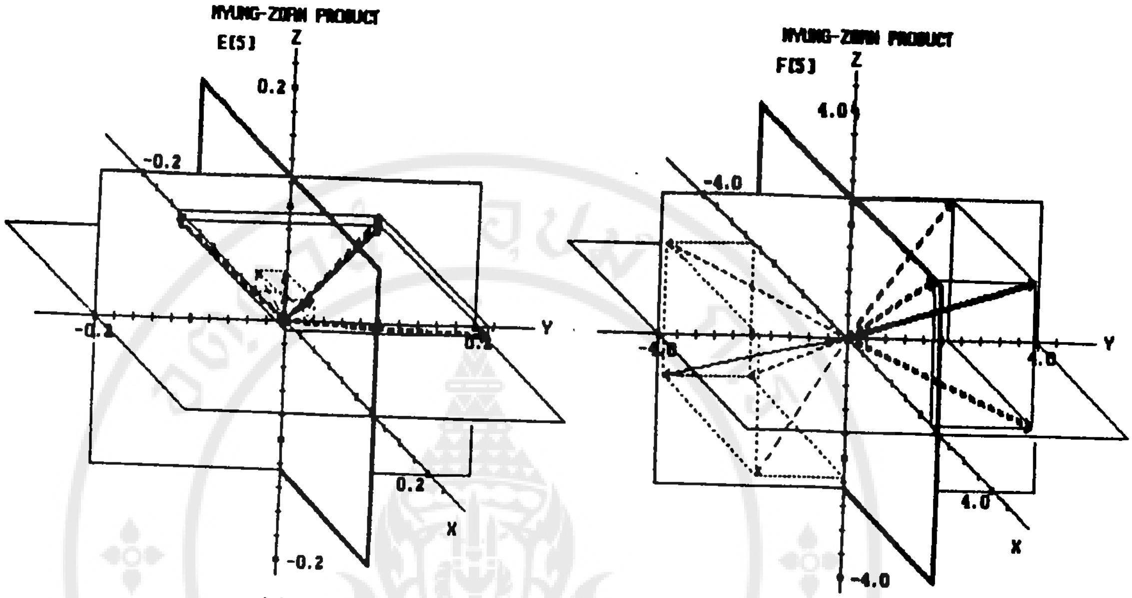
(b)


ZORN PRODUCT
C[5]
Z
2.0
-2.0
Y
-2.0
2.0
2.0
X
-2.0

(c)

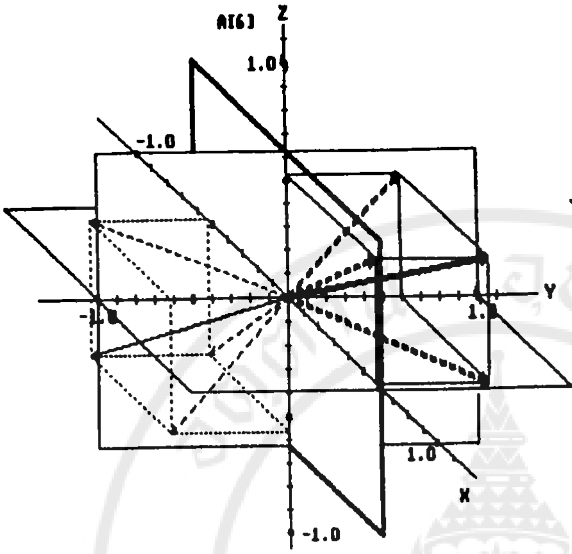

FELLOURIS PRODUCT
D[5]
Z
2.0
-2.0
Y
-2.0
2.0
2.0
X
-2.0

(d)

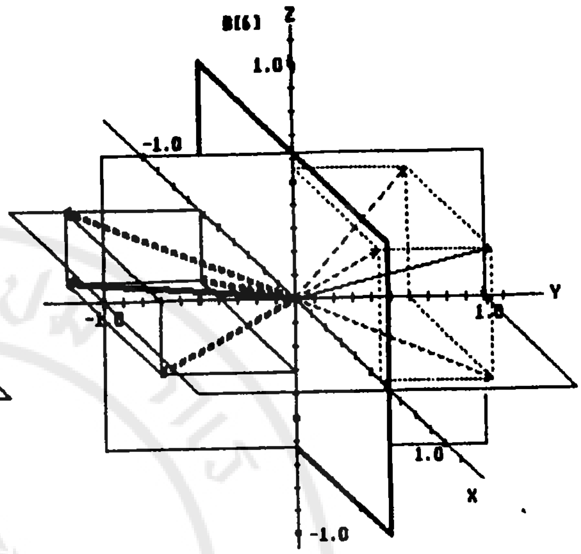


(e) (f)

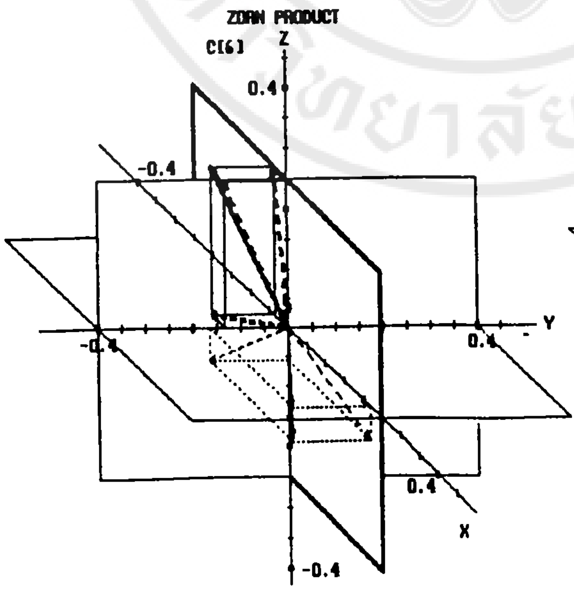
Fig. 4I(5) The representations of the vector matrices A , B , ZORN PRODUCT C , FELLOURIS PRODUCT D , MYUNG-ZORN PRODUCT E ($\tau = 0$) , and MYUNG-ZORN PRODUCT F ($\tau = \sqrt{a^2 + b^2 + |U^2| + |V^2|}$) , where A , B , C , D , E , and F $\in$ M (M = $\begin{bmatrix} a & U \\ V & b \end{bmatrix}$) , for data set 5. The scalars a and b are represented by asterisks on the Y and Z axes. The vectors U and V are represented by bold lines and ordinary lines, respectively.


A[6]
Z
1.0
-1.0
Y
-1.0
1.0
1.0
X
-1.0

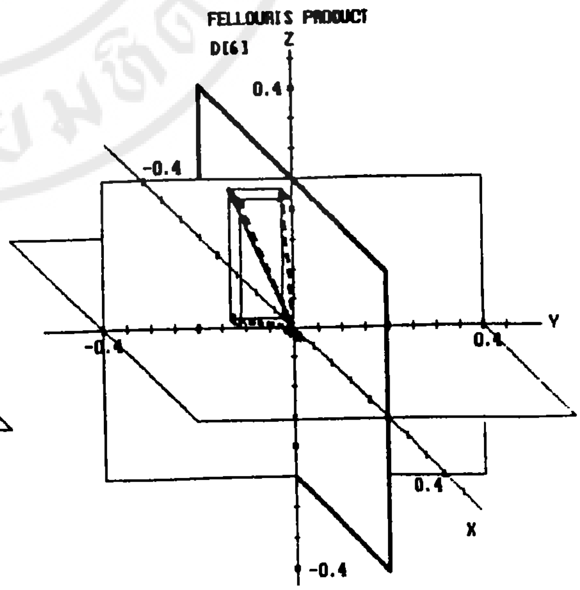
(a)


B[6]
Z
1.0
-1.0
Y
-1.0
1.0
1.0
X
-1.0

(b)


ZORN PRODUCT
C[6]
Z
0.4
-0.4
Y
-0.4
0.4
0.4
X
-0.4

(c)


FELLOURIS PRODUCT
D[6]
Z
0.4
-0.4
Y
-0.4
0.4
0.4
X
-0.4

(d)

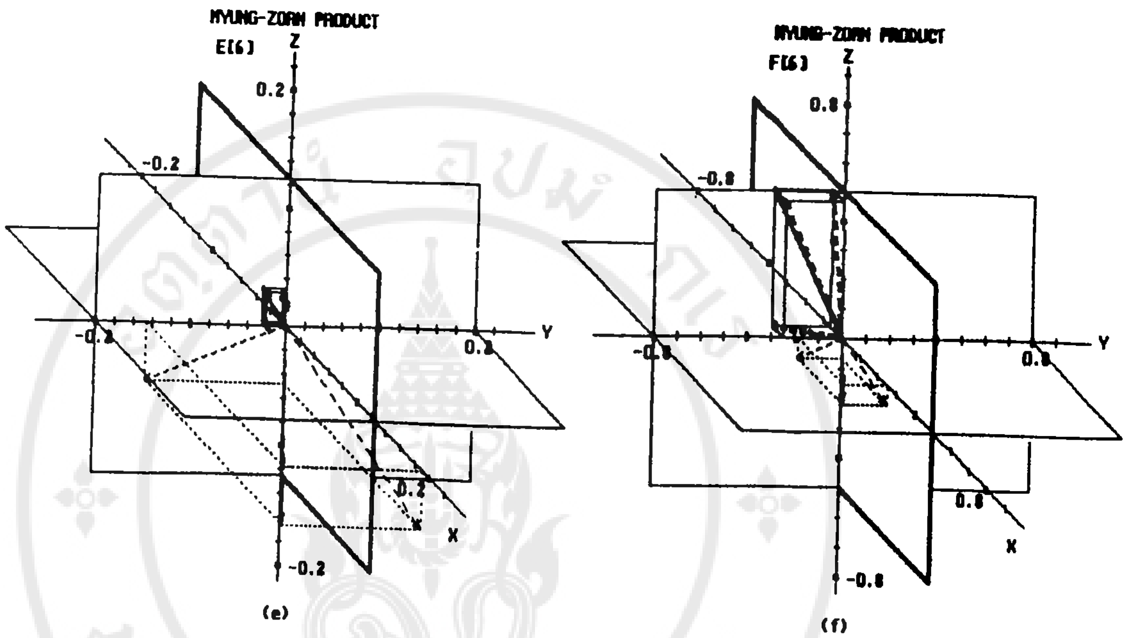
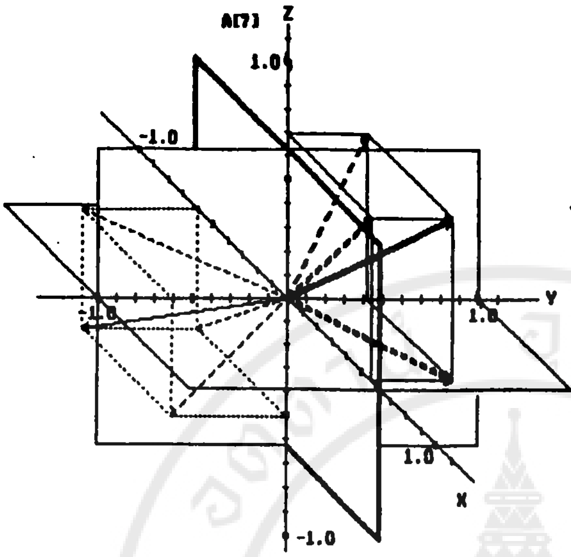
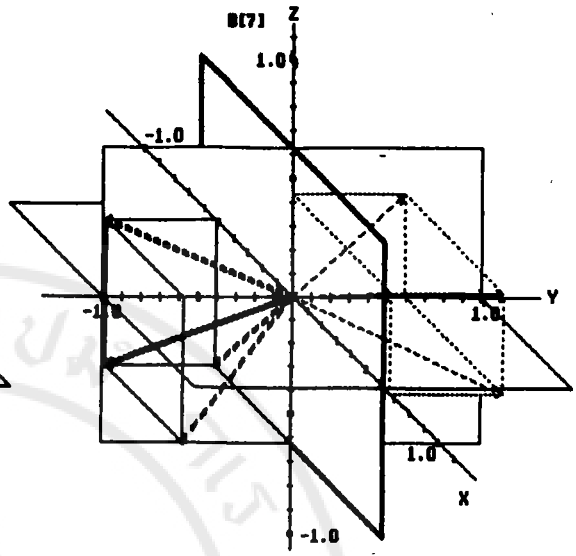


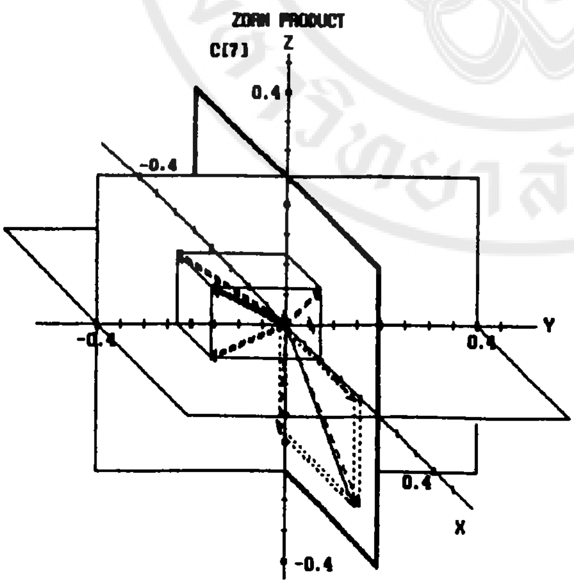
Fig. 4I(6) The representations of the vector matrices $\mathbf{A}$, $\mathbf{B}$, ZORN PRODUCT $\mathbf{C}$, FELLOURIS PRODUCT $\mathbf{D}$, MYUNG-ZORN PRODUCT $\mathbf{E}$ ($\tau = 0$) , and MYUNG-ZORN PRODUCT $\mathbf{F}$ ($\tau = \sqrt{a^2 + b^2 + |U^2| + |V^2|}$) , where $\mathbf{A}$, $\mathbf{B}$, $\mathbf{C}$, $\mathbf{D}$, $\mathbf{E}$, and $\mathbf{F} \in \mathbf{M}$ ($\mathbf{M} = \begin{bmatrix} a & U \\ V & b \end{bmatrix}$) , for data set 6. The scalars a and b are represented by asterisks on the Y and Z axes. The vectors U and V are represented by bold lines and ordinary lines, respectively.


A[7]
Z
1.0
-1.0
Y
1.0
X
-1.0

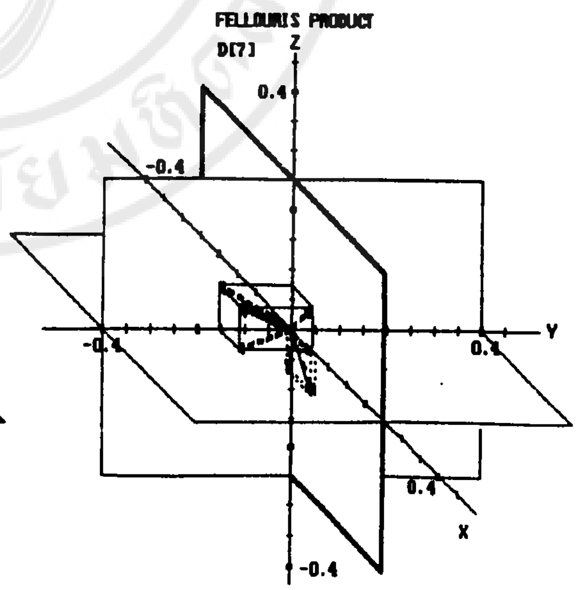
(a)


B[7]
Z
1.0
-1.0
Y
1.0
X
-1.0

(b)


ZORN PRODUCT
C[7]
Z
0.4
-0.4
Y
0.4
X
-0.4

(c)


FELLOURIS PRODUCT
D[7]
Z
0.4
-0.4
Y
0.4
X
-0.4

(d)

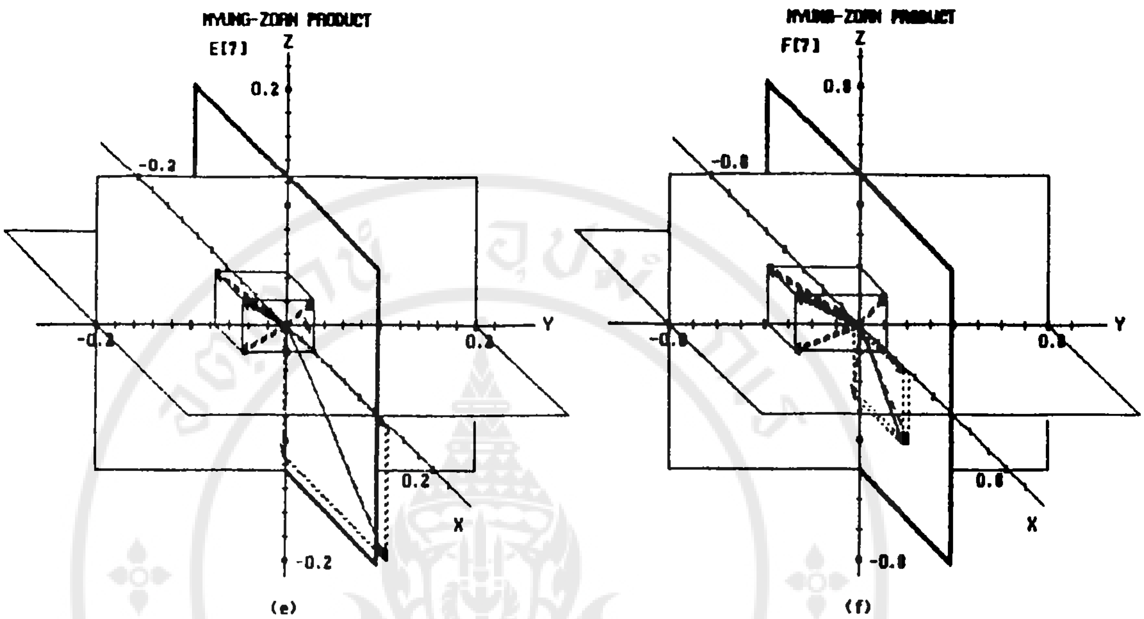
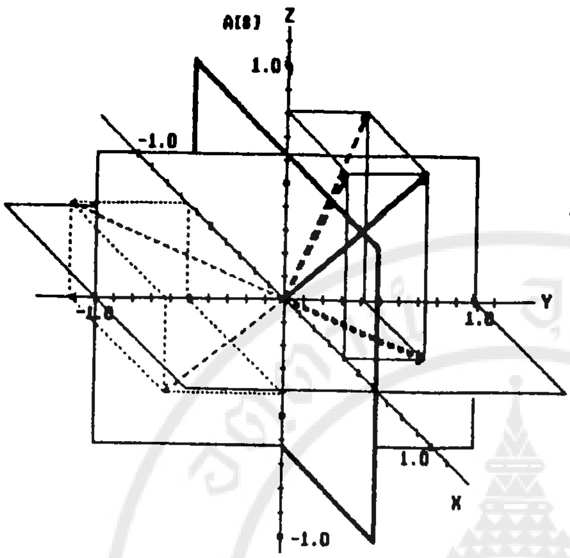
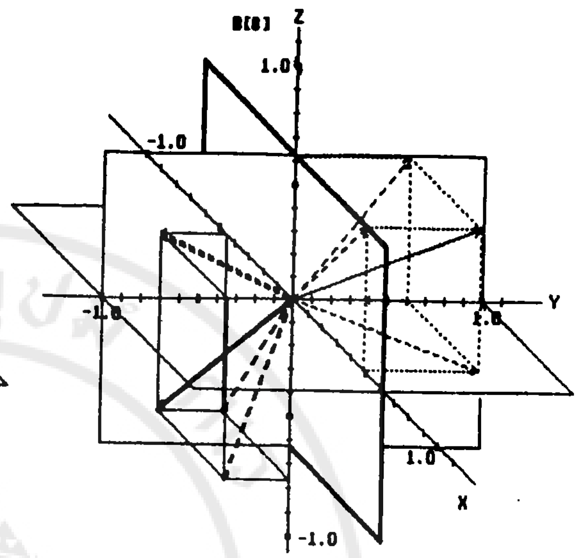


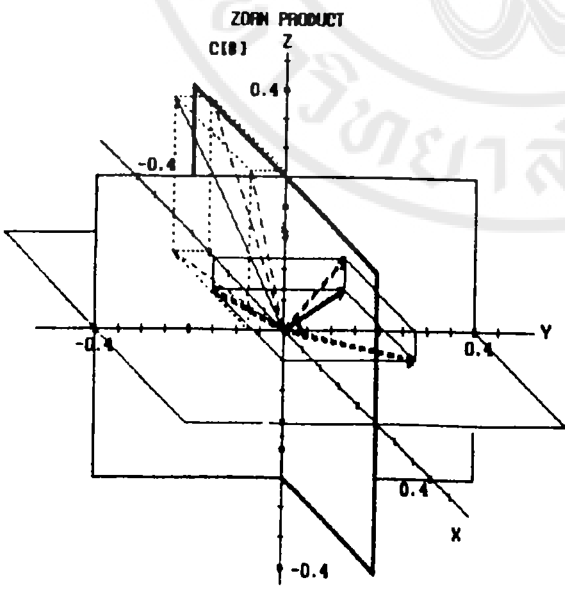
Fig. 41(7) The representations of the vector matrices A, B, ZORN PRODUCT C, FELLOURIS PRODUCT D, MYUNG-ZORN PRODUCT E ($\tau = 0$), and MYUNG-ZORN PRODUCT F ($\tau = \sqrt{a^2 + b^2 + |U^2| + |V^2|}$), where A, B, C, D, E, and F $\in$ M ($M = \begin{bmatrix} a & U \\ V & b \end{bmatrix}$), for data set 7. The scalars a and b are represented by asterisks on the Y and Z axes. The vectors U and V are represented by bold lines and ordinary lines, respectively.


A[8]
Z
1.0
-1.0
-1.0
Y
1.0
1.0
X
-1.0

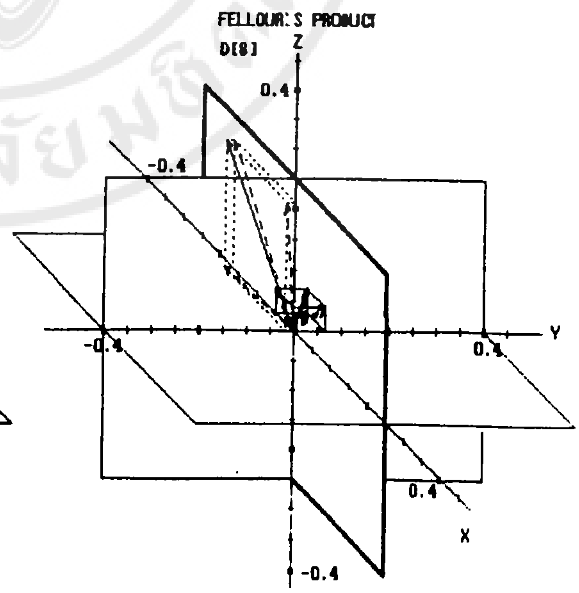
(a)


B[8]
Z
1.0
-1.0
-1.0
Y
1.0
1.0
X
-1.0

(b)


ZORN PRODUCT
C[8]
Z
0.4
-0.4
-0.4
Y
0.4
0.4
X
-0.4

(c)


FELLOUR'S PRODUCT
D[8]
Z
0.4
-0.4
-0.4
Y
0.4
0.4
X
-0.4

(d)

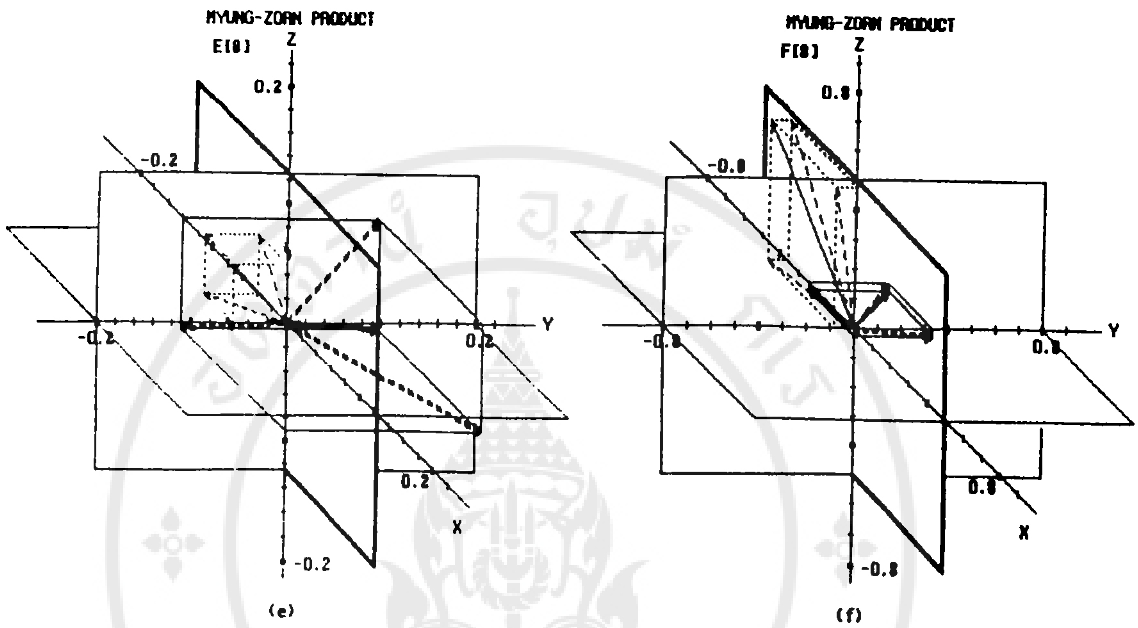
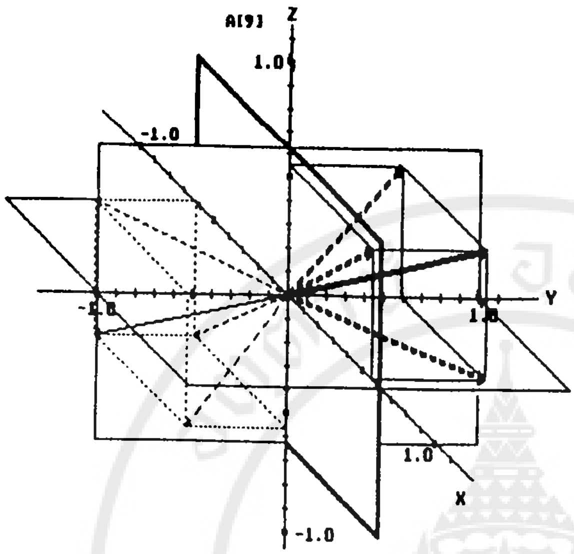
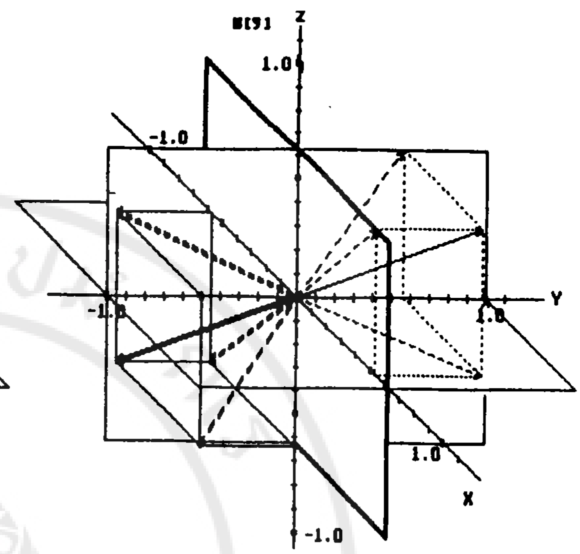


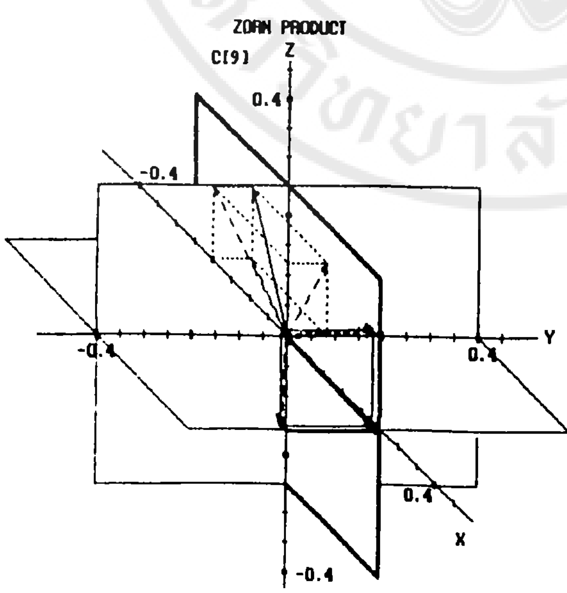
Fig. 41(8) The representations of the vector matrices A , B , ZORN PRODUCT C , FELLOURIS PRODUCT D , MYUNG-ZORN PRODUCT E ($\tau = 0$) , and MYUNG-ZORN PRODUCT F ($\tau = \sqrt{a^2 + b^2 + |U^2| + |V^2|}$) , where A , B , C , D , E , and F $\in$ M ($M = \begin{bmatrix} a & U \\ V & b \end{bmatrix}$) , for data set 8. The scalars a and b are represented by asterisks on the Y and Z axes. The vectors U and V are represented by bold lines and ordinary lines, respectively.


A[9]
Z
1.0
-1.0
Y
1.0
X
-1.0

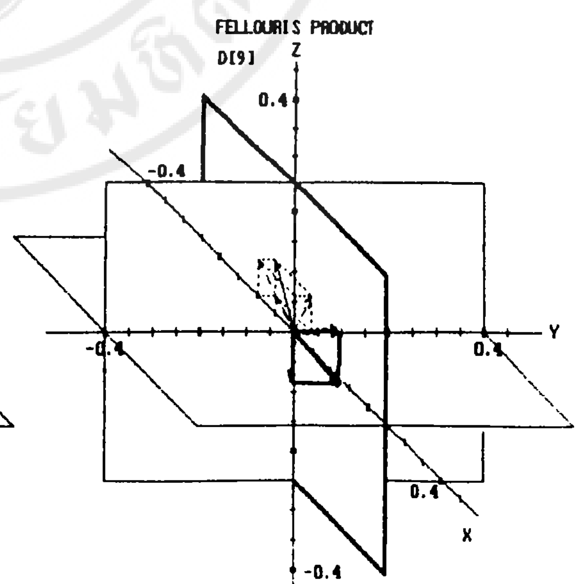
(a)


Z
1.0
-1.0
Y
1.0
X
-1.0

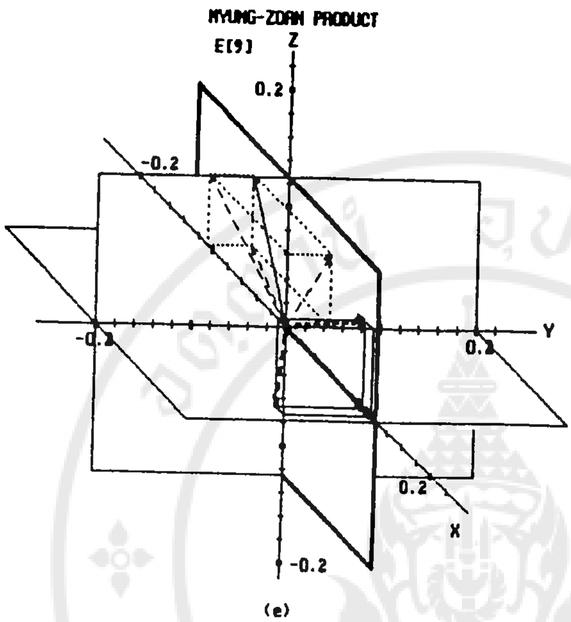
(b)


ZORN PRODUCT
C[9]
Z
0.4
-0.4
Y
0.4
X
-0.4

(c)


FELLOURIS PRODUCT
D[9]
Z
0.4
-0.4
Y
0.4
X
-0.4

(d)



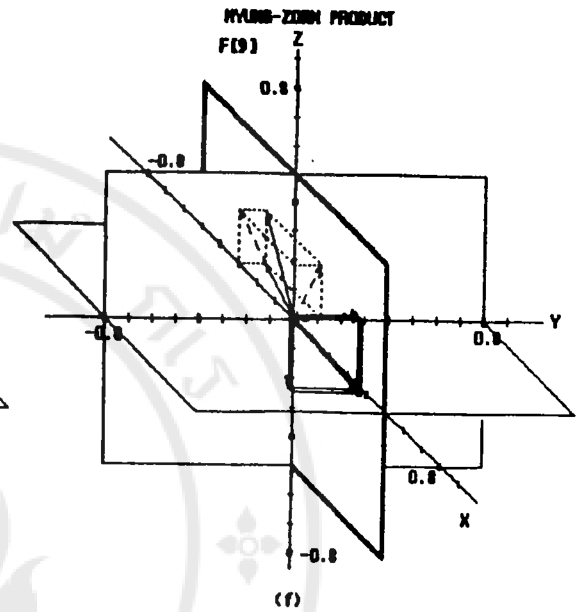
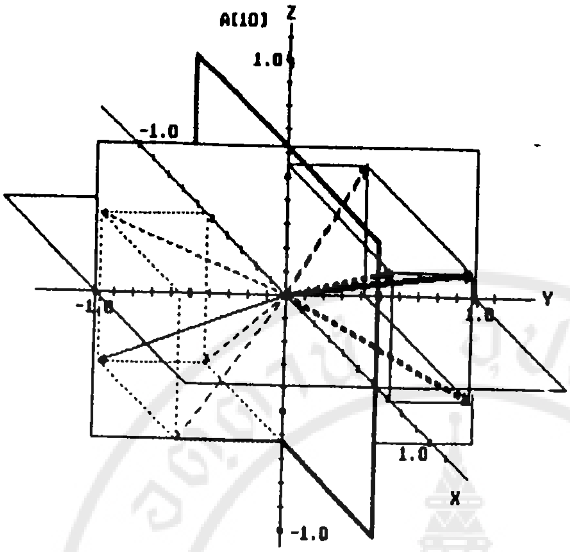
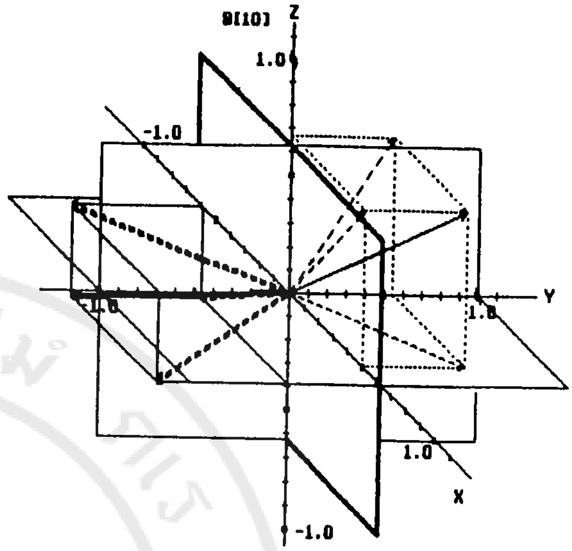


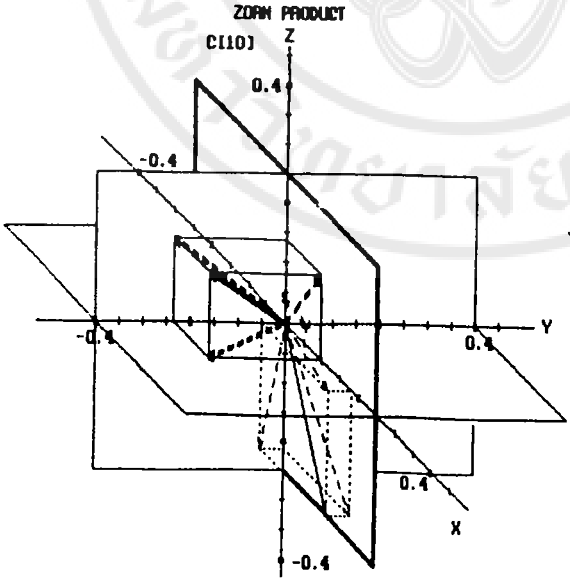
Fig. 4I(9) The representations of the vector matrices A, B, ZORN PRODUCT C, FELLOURIS PRODUCT D, MYUNG-ZORN PRODUCT E ($\tau = 0$), and MYUNG-ZORN PRODUCT F ($\tau = \sqrt{a^2 + b^2 + |U^2| + |V^2|}$), where A, B, C, D, E, and F $\in$ M ($M = \begin{bmatrix} a & U \\ V & b \end{bmatrix}$), for data set 9. The scalars a and b are represented by asterisks on the Y and Z axes. The vectors U and V are represented by bold lines and ordinary lines, respectively.


A[10]
Z
1.0
-1.0
-1.0
1.0
Y
1.0
X
-1.0

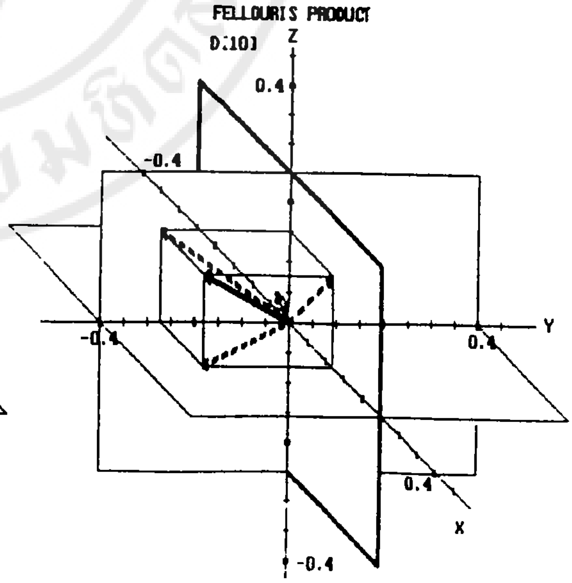
(a)


B[10]
Z
1.0
-1.0
-1.0
1.0
Y
1.0
X
-1.0

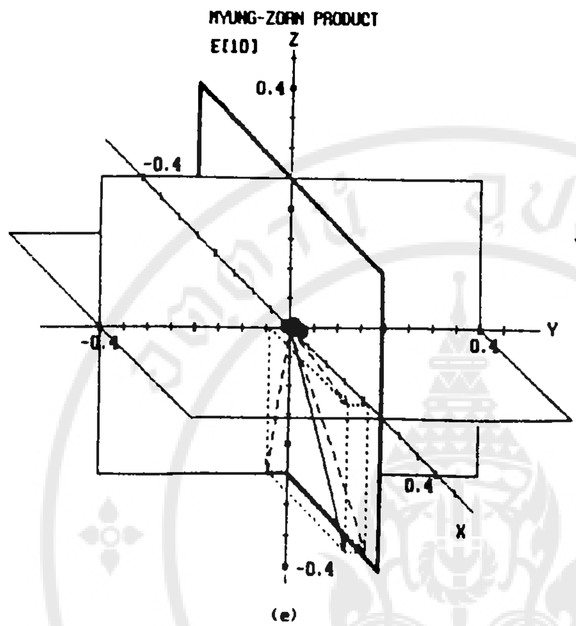
(b)


ZORN PRODUCT
C[10]
Z
0.4
-0.4
-0.4
0.4
Y
0.4
X
-0.4

(c)


FELLOURIS PRODUCT
D[10]
Z
0.4
-0.4
-0.4
0.4
Y
0.4
X
-0.4

(d)



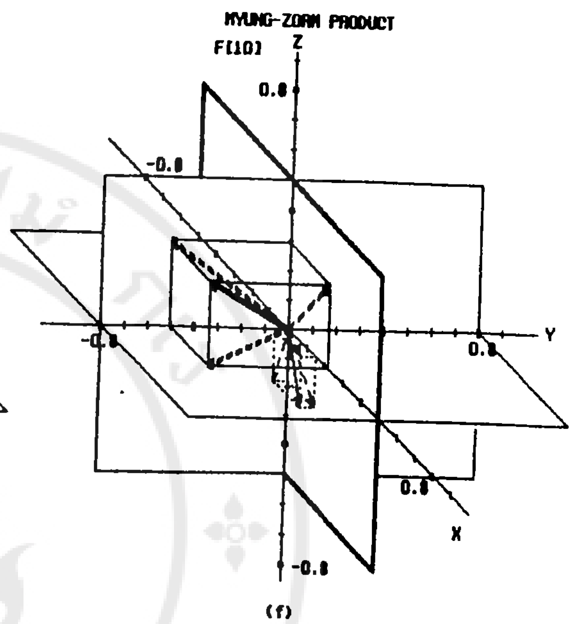
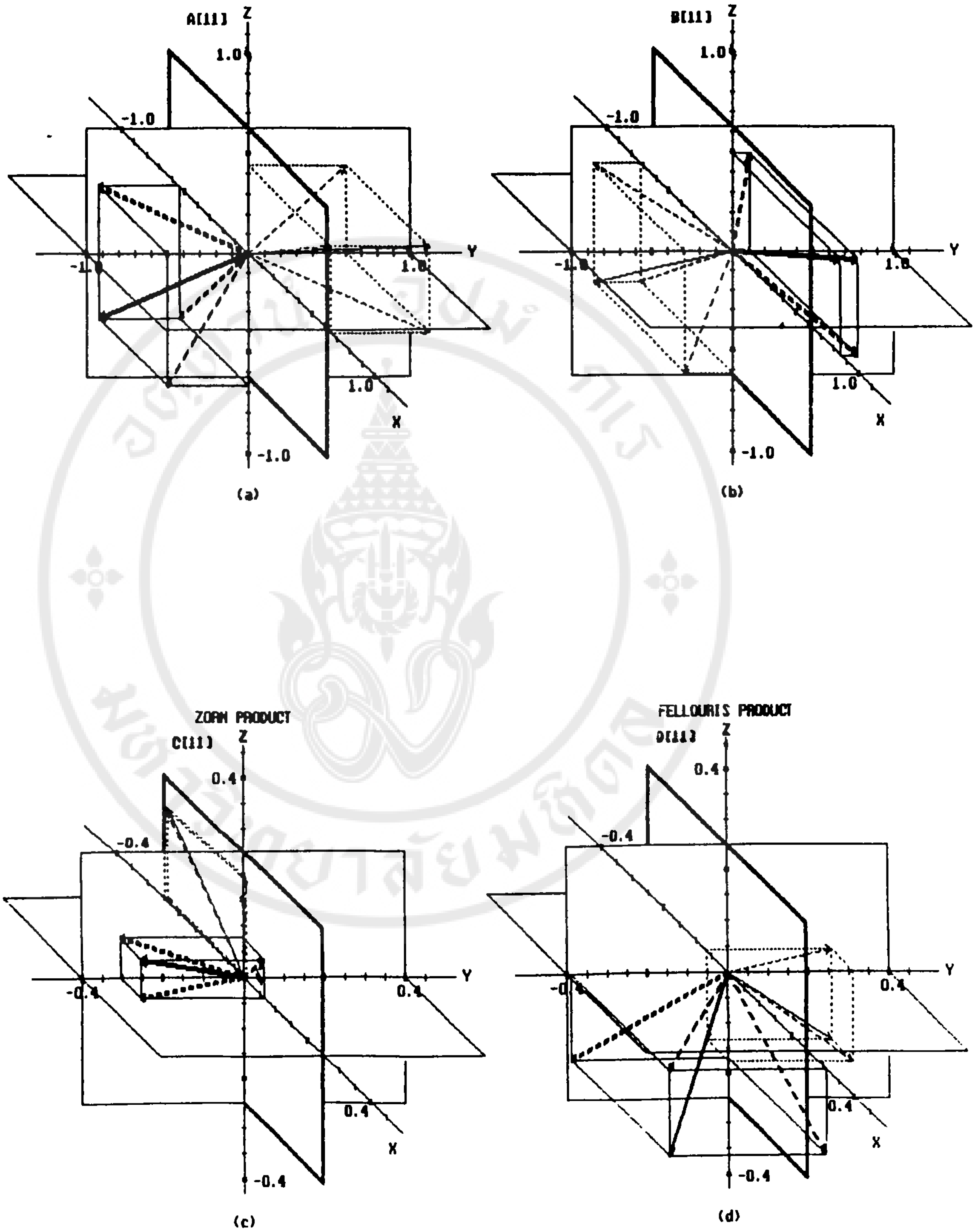
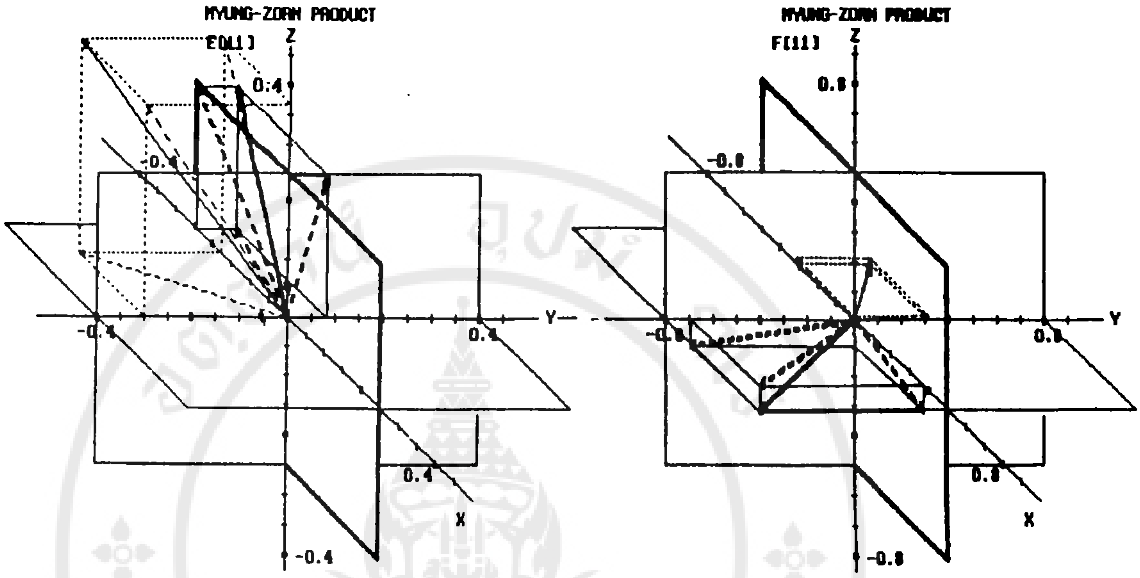


Fig. 41(10) The representations of the vector matrices A, B, ZORN PRODUCT C, FELLOURIS PRODUCT D, MYUNG-ZORN PRODUCT E ($\tau = 0$), and MYUNG-ZORN PRODUCT F ($\tau = \sqrt{a^2 + b^2 + |U^2| + |V^2|}$), where A, B, C, D, E, and F $\in$ M ($M = \begin{bmatrix} a & U \\ V & b \end{bmatrix}$), for data set 10. The scalars a and b are represented by asterisks on the Y and Z axes. The vectors U and V are represented by bold lines and ordinary lines, respectively.

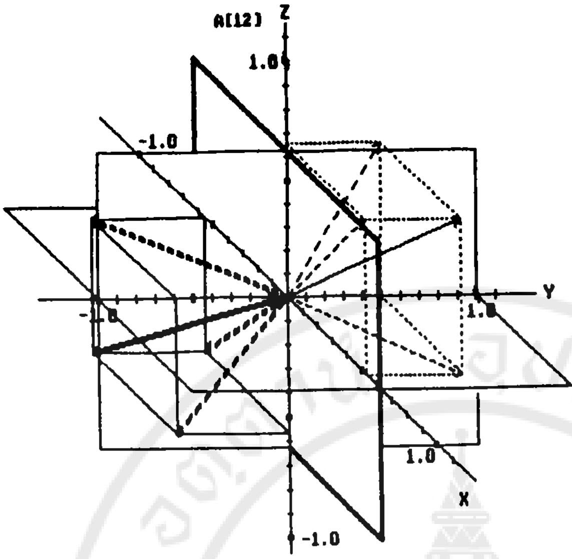

A[11]
Z
1.0
-1.0
Y
X
B[11]
ZORN PRODUCT
C[11]
0.4
-0.4
FELLOURIS PRODUCT
D[11]

(a) (b) (c) (d)

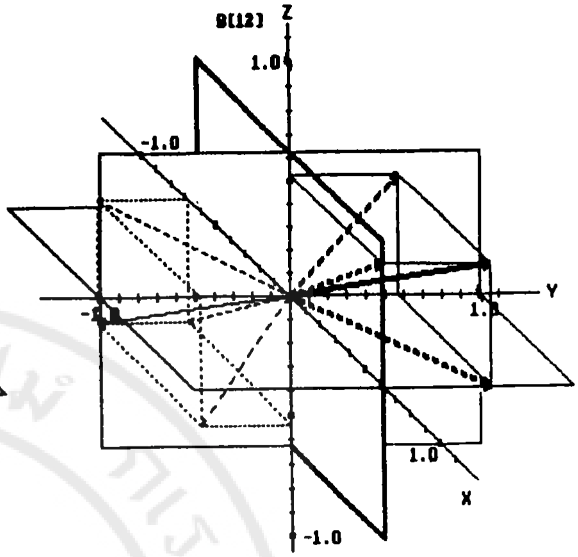


(e) (f)

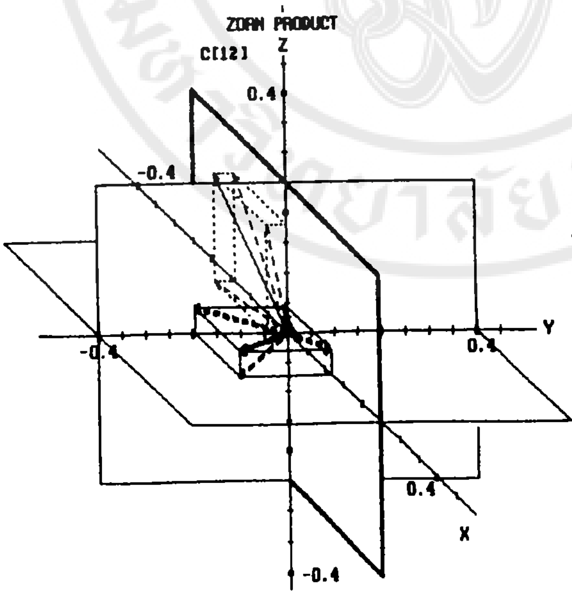
Fig. 4I(11) The representations of the vector matrices $\mathbf{A}$, $\mathbf{B}$, ZORN PRODUCT $\mathbf{C}$, FELLOURIS PRODUCT $\mathbf{D}$, MYUNG-ZORN PRODUCT $\mathbf{E}$ ($\tau = 0$) , and MYUNG-ZORN PRODUCT $\mathbf{F}$ ($\tau = \sqrt{a^2 + b^2 + |U^2| + |V^2|}$) , where $\mathbf{A}$, $\mathbf{B}$, $\mathbf{C}$, $\mathbf{D}$, $\mathbf{E}$, and $\mathbf{F} \in \mathbf{M}$ ($\mathbf{M} = \begin{bmatrix} a & U \\ V & b \end{bmatrix}$) , for data set 11. The scalars a and b are represented by asterisks on the Y and Z axes. The vectors U and V are represented by bold lines and ordinary lines, respectively.


A[12]
Z
Y
X
1.0
-1.0
1.0
1.0
-1.0

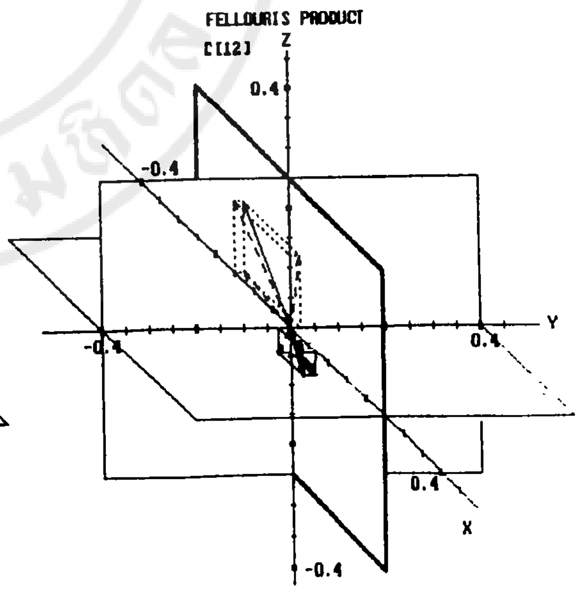
(a)


B[12]
Z
Y
X
1.0
-1.0
1.0
-1.0

(b)


ZORN PRODUCT
C[12]
Z
Y
X
0.4
-0.4
0.4
0.4
-0.4

(c)


FELLOURIS PRODUCT
C[12]
Z
Y
X
0.4
-0.4
0.4
0.4
-0.4

(d)

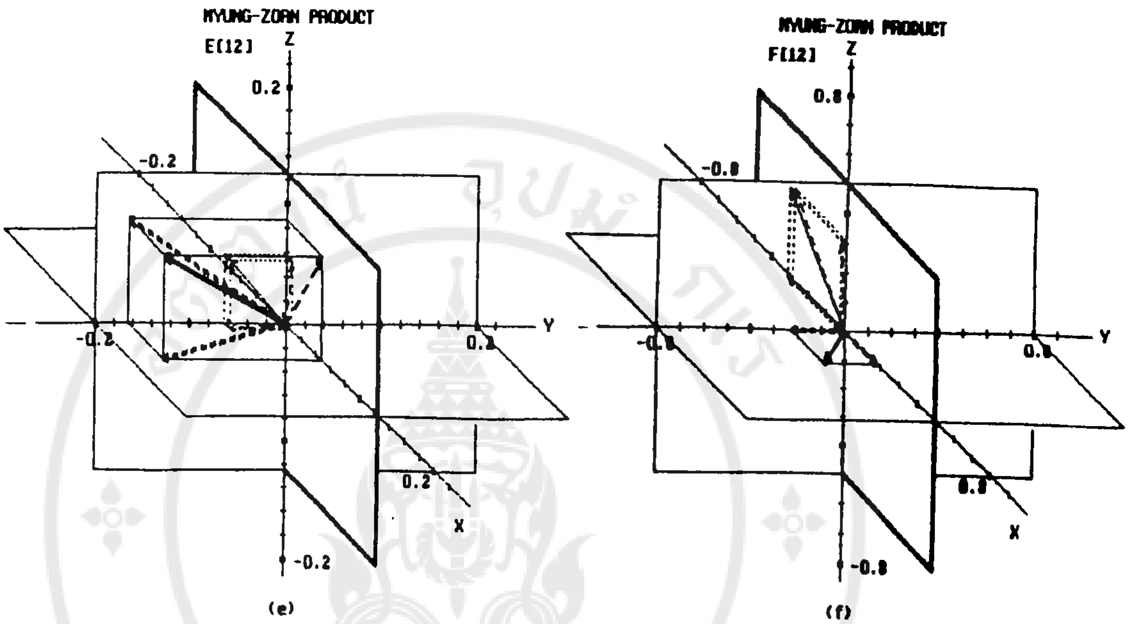
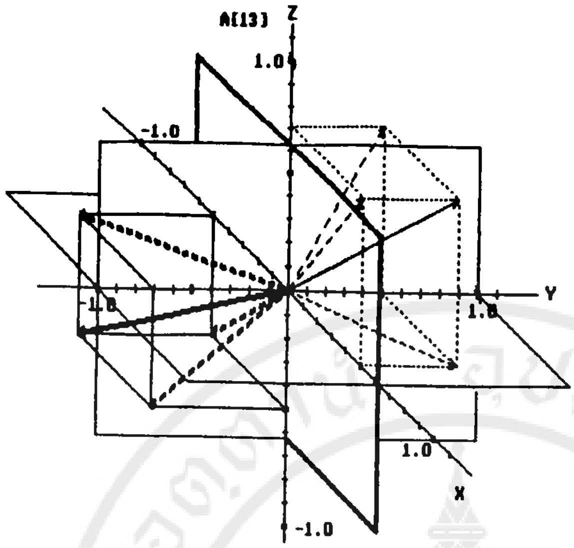
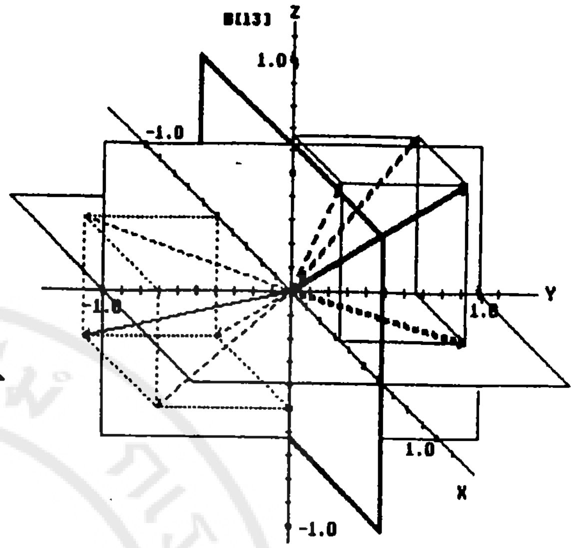


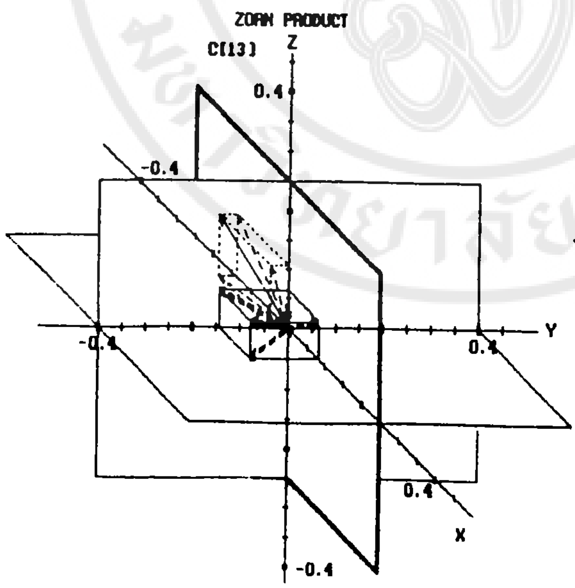
Fig. 4I(12) The representations of the vector matrices A, B, ZORN PRODUCT C; FELLOURIS PRODUCT D, MYUNG-ZORN PRODUCT E ($\tau = 0$), and MYUNG-ZORN PRODUCT F ($\tau = \sqrt{a^2 + b^2 + |U^2| + |V^2|}$), where A, B, C, D, E, and F $\in$ M (M = $\begin{bmatrix} a & U \\ V & b \end{bmatrix}$), for data set 12. The scalars a and b are represented by asterisks on the Y and Z axes. The vectors U and V are represented by bold lines and ordinary lines, respectively.


A[13]
Z
1.0
-1.0
Y
-1.0
1.0
1.0
X
-1.0

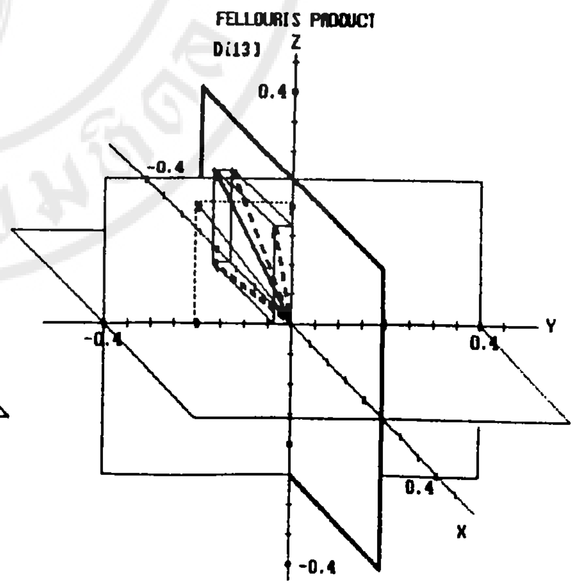
(a)


B[13]
Z
1.0
-1.0
Y
-1.0
1.0
1.0
X
-1.0

(b)


ZORN PRODUCT
C[13]
Z
0.4
-0.4
Y
-0.4
0.4
0.4
X
-0.4

(c)


FELLOURIS PRODUCT
D[13]
Z
0.4
-0.4
Y
-0.4
0.4
0.4
X
-0.4

(d)

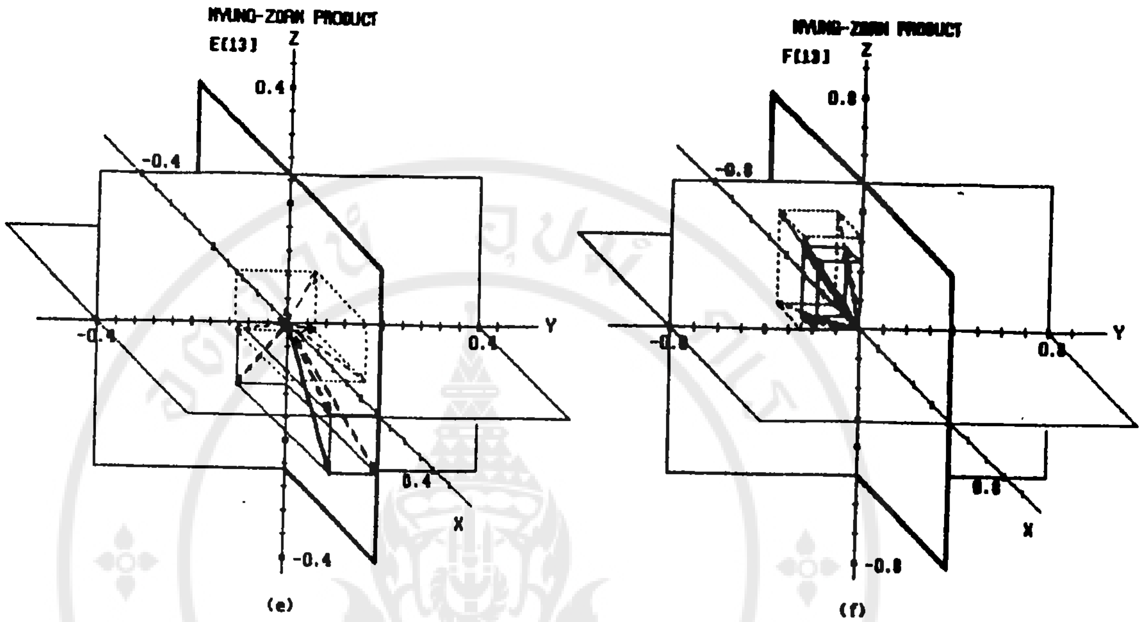
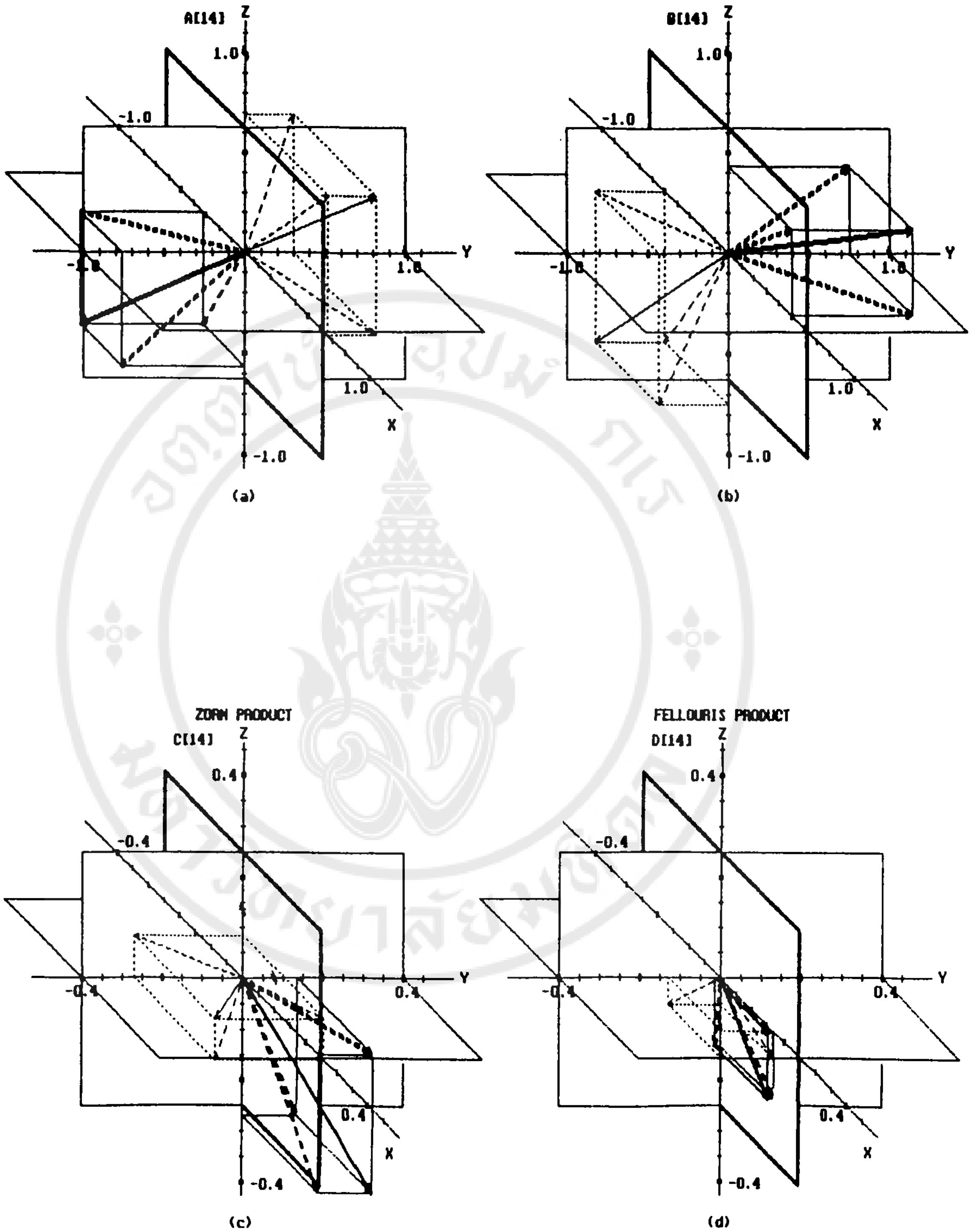


Fig. 4I(13) The representations of the vector matrices A , B , ZORN PRODUCT C , FELLOURIS PRODUCT D , MYUNG-ZORN PRODUCT E ($\tau = 0$) , and MYUNG-ZORN PRODUCT F ($\tau = \sqrt{a^2 + b^2 + |U^2| + |V^2|}$) , where A , B , C , D , E , and F $\in$ M ($M = \begin{bmatrix} a & U \\ V & b \end{bmatrix}$) , for data set 13. The scalars a and b are represented by asterisks on the Y and Z axes. The vectors U and V are represented by bold lines and ordinary lines, respectively.


A[14]
Z
1.0
-1.0
Y
1.0
X
-1.0
(a)
B[14]
Z
1.0
-1.0
Y
1.0
X
-1.0
(b)
ZORN PRODUCT
C[14]
Z
0.4
-0.4
Y
0.4
X
-0.4
(c)
FELLOURIS PRODUCT
D[14]
Z
0.4
-0.4
Y
0.4
X
-0.4
(d)

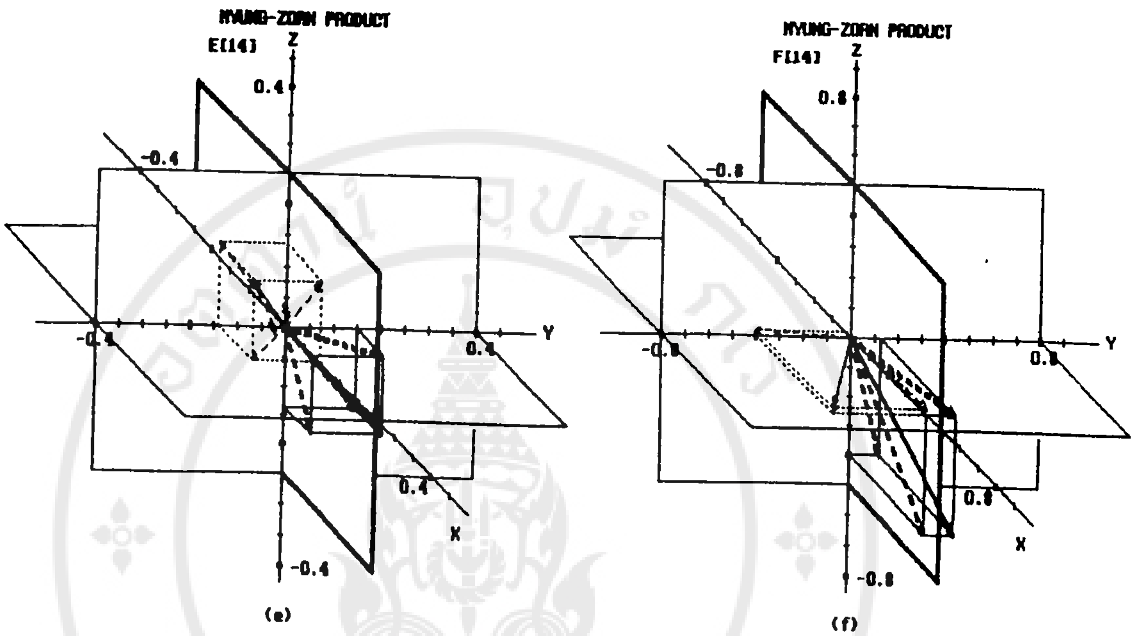
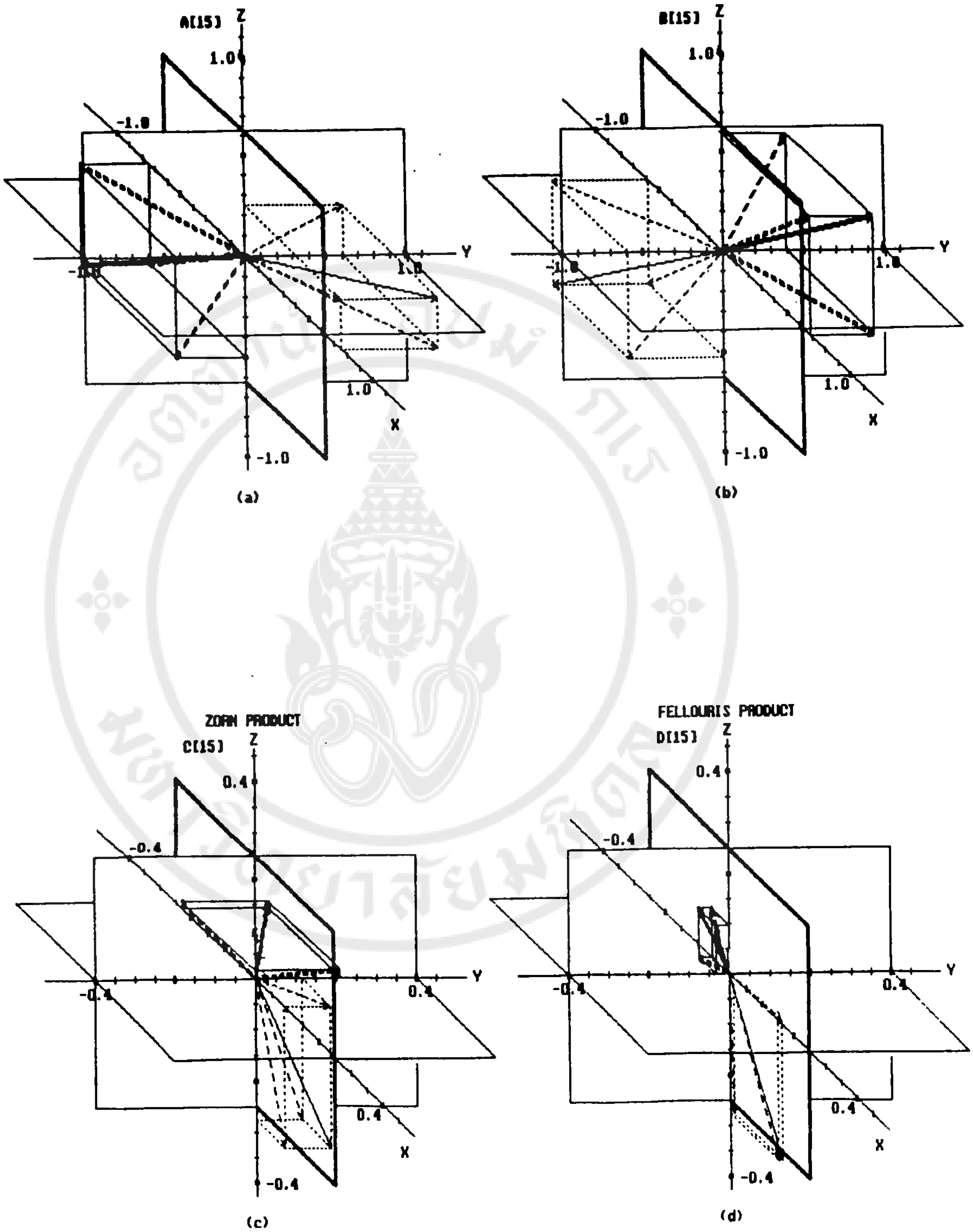


Fig. 41(14) The representations of the vector matrices **A** , **B** , ZORN PRODUCT **C** , FELLOURIS PRODUCT **D** , MYUNG-ZORN PRODUCT **E** ($\tau = 0$) , and MYUNG-ZORN PRODUCT **F** ($\tau = \sqrt{a^2 + b^2 + |U^2| + |V^2|}$) , where **A** , **B** , **C** , **D** , **E** , and **F** $\in$ **M** (**M** = $\begin{bmatrix} a & U \\ V & b \end{bmatrix}$) , for data set 14. The scalars a and b are represented by asterisks on the Y and Z axes. The vectors U and V are represented by bold lines and ordinary lines, respectively.


A[15]
Z
1.0
-1.0
Y
X
(a)
B[15]
(b)
ZORN PRODUCT
C[15]
0.4
-0.4
(c)
FELLOURIS PRODUCT
D[15]
(d)

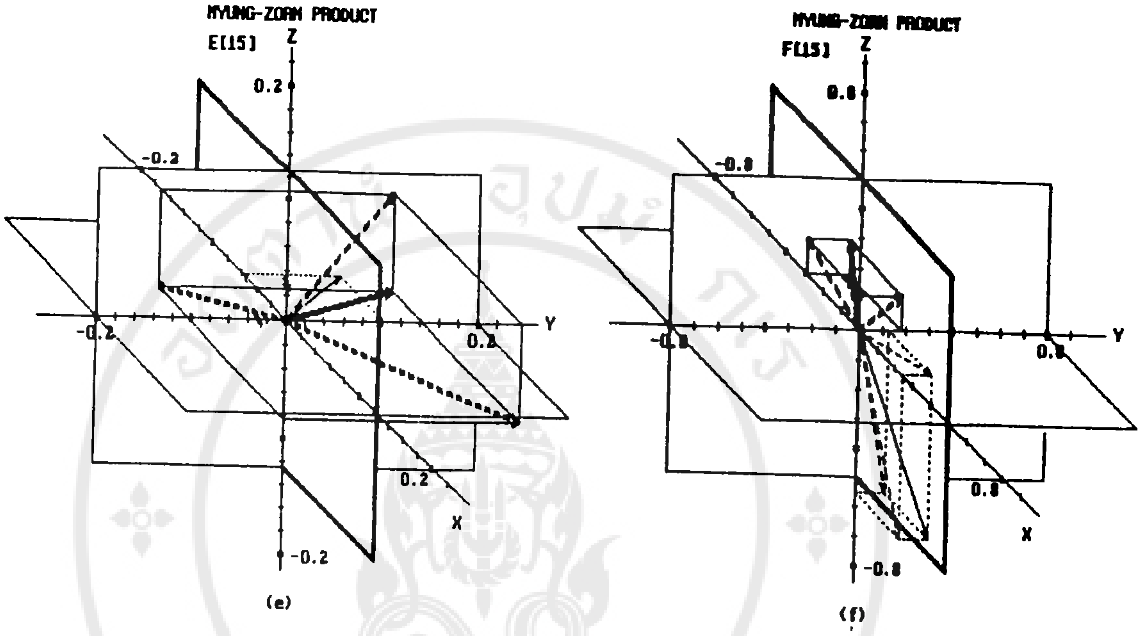
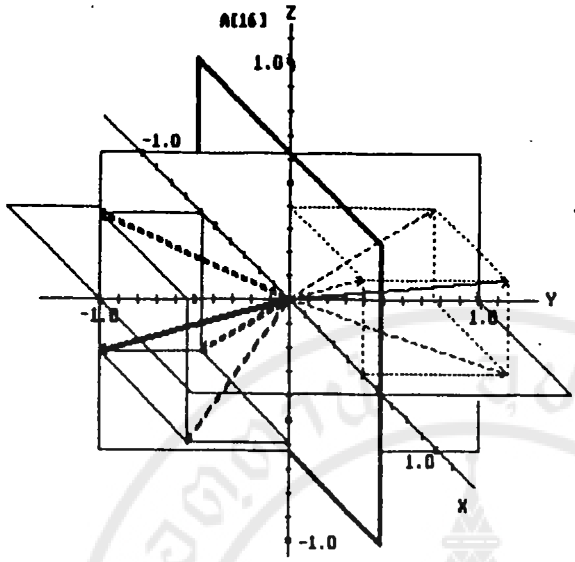
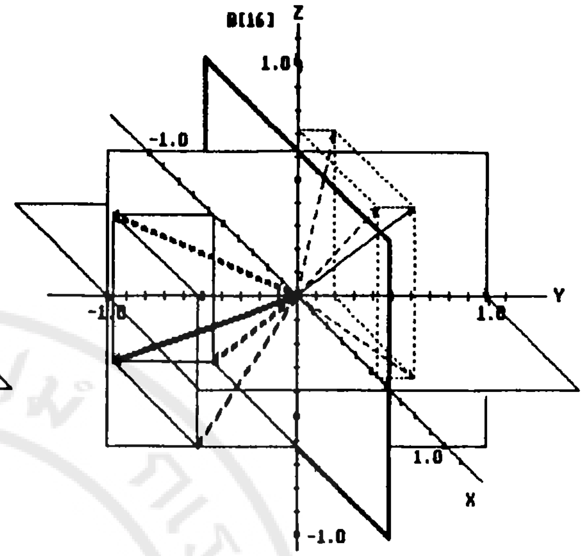


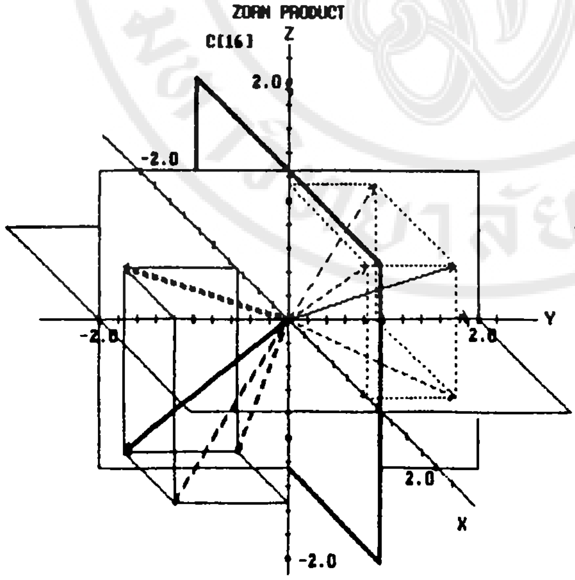
Fig. 41(15) The representations of the vector matrices A , B , ZORN PRODUCT C , FELLOURIS PRODUCT D , MYUNG-ZORN PRODUCT E ($\tau = 0$) , and MYUNG-ZORN PRODUCT F ($\tau = \sqrt{a^2 + b^2 + |U^2| + |V^2|}$) , where A , B , C , D , E , and F $\in$ M ($M = \begin{bmatrix} a & U \\ V & b \end{bmatrix}$) , for data set 15. The scalars a and b are represented by asterisks on the Y and Z axes. The vectors U and V are represented by bold lines and ordinary lines, respectively.


A[16]
Z
1.0
-1.0
Y
-1.0
1.0
1.0
X
-1.0

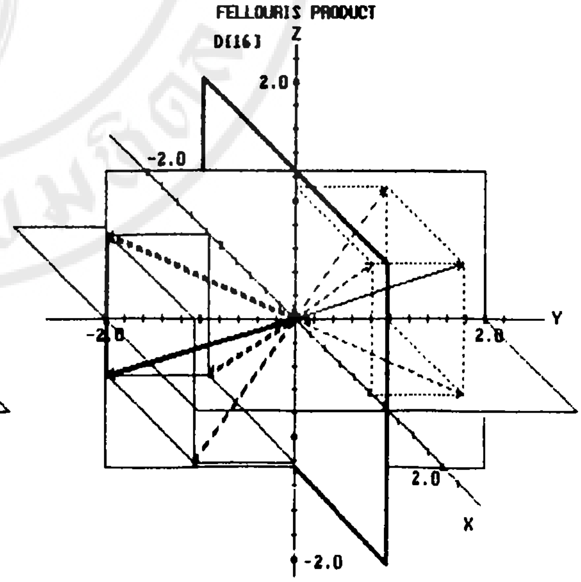
(a)


B[16]
Z
1.0
-1.0
Y
-1.0
1.0
1.0
X
-1.0

(b)


ZORN PRODUCT
C[16]
Z
2.0
-2.0
Y
-2.0
2.0
2.0
X
-2.0

(c)


FELLOURIS PRODUCT
D[16]
Z
2.0
-2.0
Y
-2.0
2.0
2.0
X
-2.0

(d)

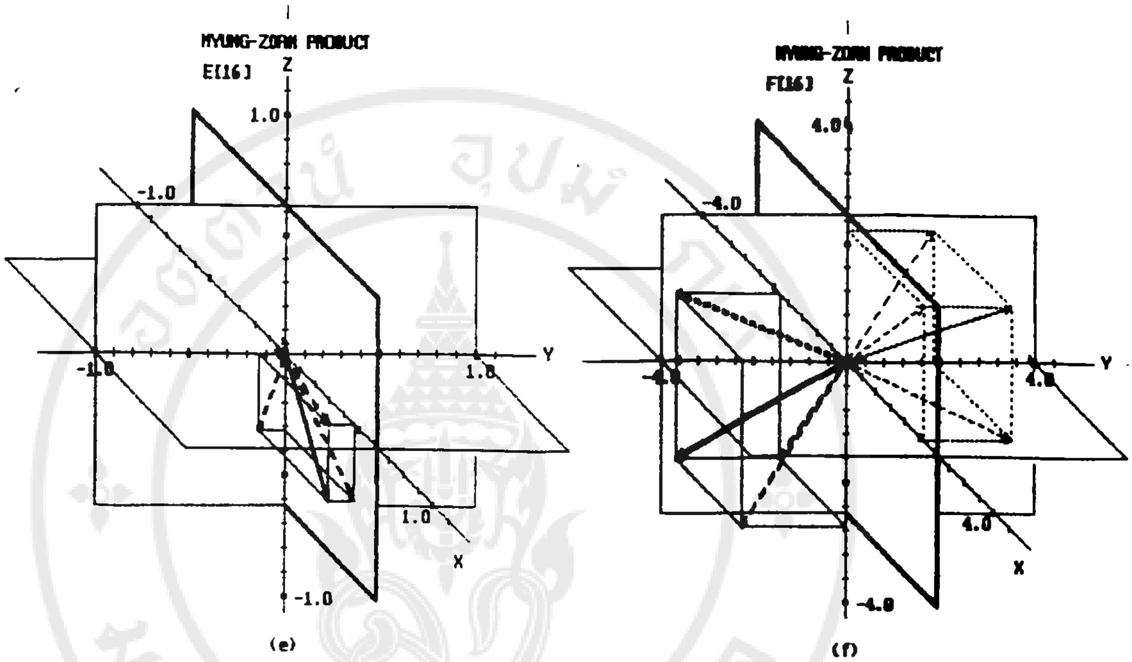
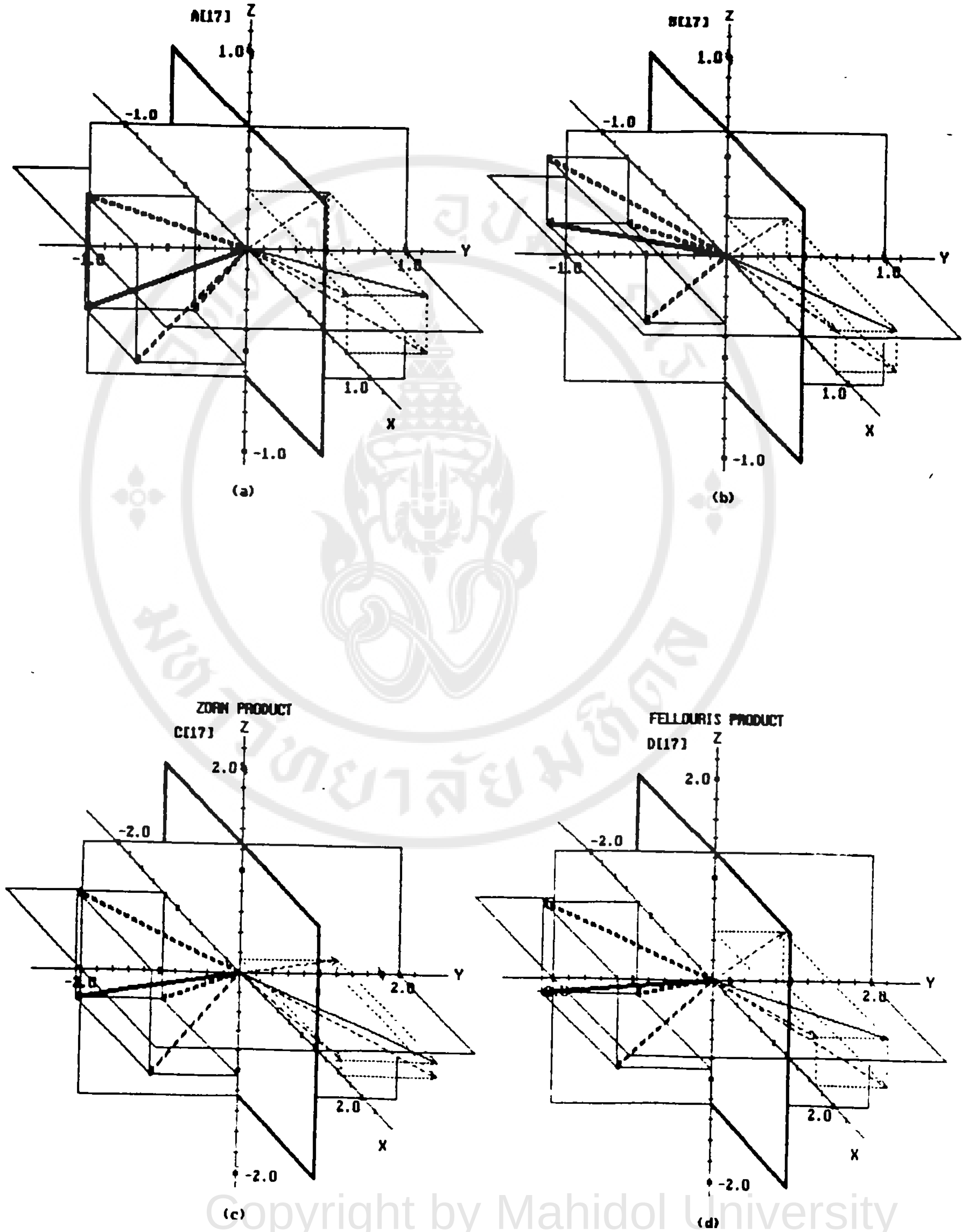


Fig. 4I(16) The representations of the vector matrices **A**, **B**, ZORN PRODUCT **C**, FELLOURIS PRODUCT **D**, MYUNG-ZORN PRODUCT **E** ($\tau = 0$) , and MYUNG-ZORN PRODUCT **F** ($\tau = \sqrt{a^2 + b^2 + |U^2| + |V^2|}$) , where **A** , **B** , **C** , **D** , **E** , and **F** $\in$ **M** (**M** = $\begin{bmatrix} a & U \\ V & b \end{bmatrix}$) , for data set 16. The scalars a and b are represented by asterisks on the Y and Z axes. The vectors U and V are represented by bold lines and ordinary lines, respectively.


A[17]
Z
1.0
-1.0
Y
X
(a)
B[17]
(b)
ZORN PRODUCT
C[17]
2.0
-2.0
(c)
FELLOURIS PRODUCT
D[17]
(d)

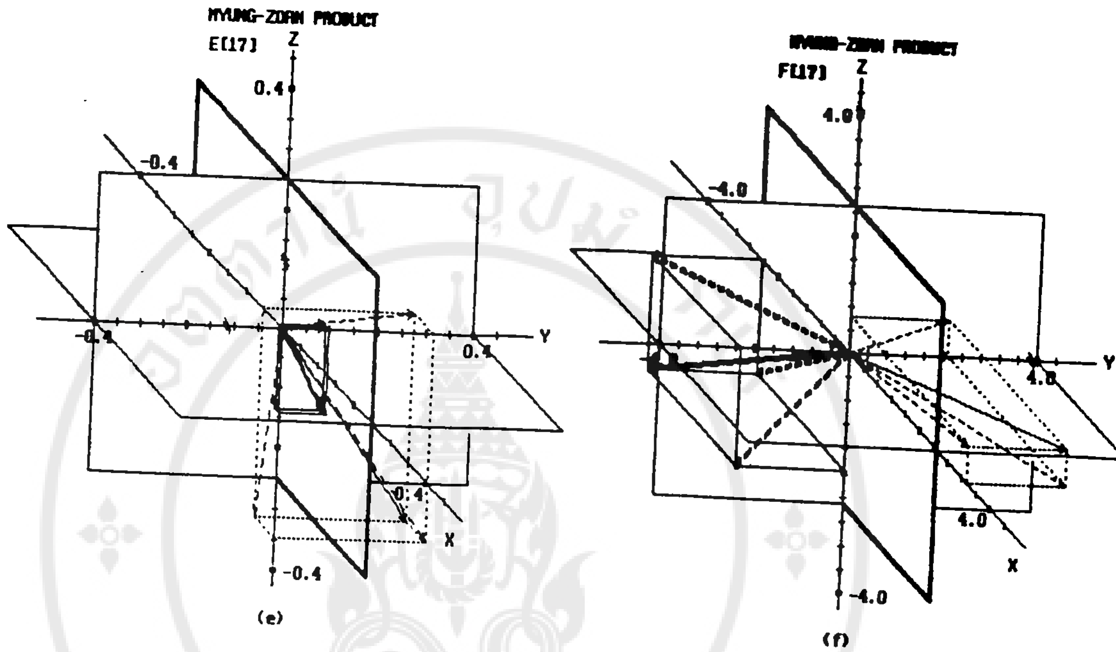
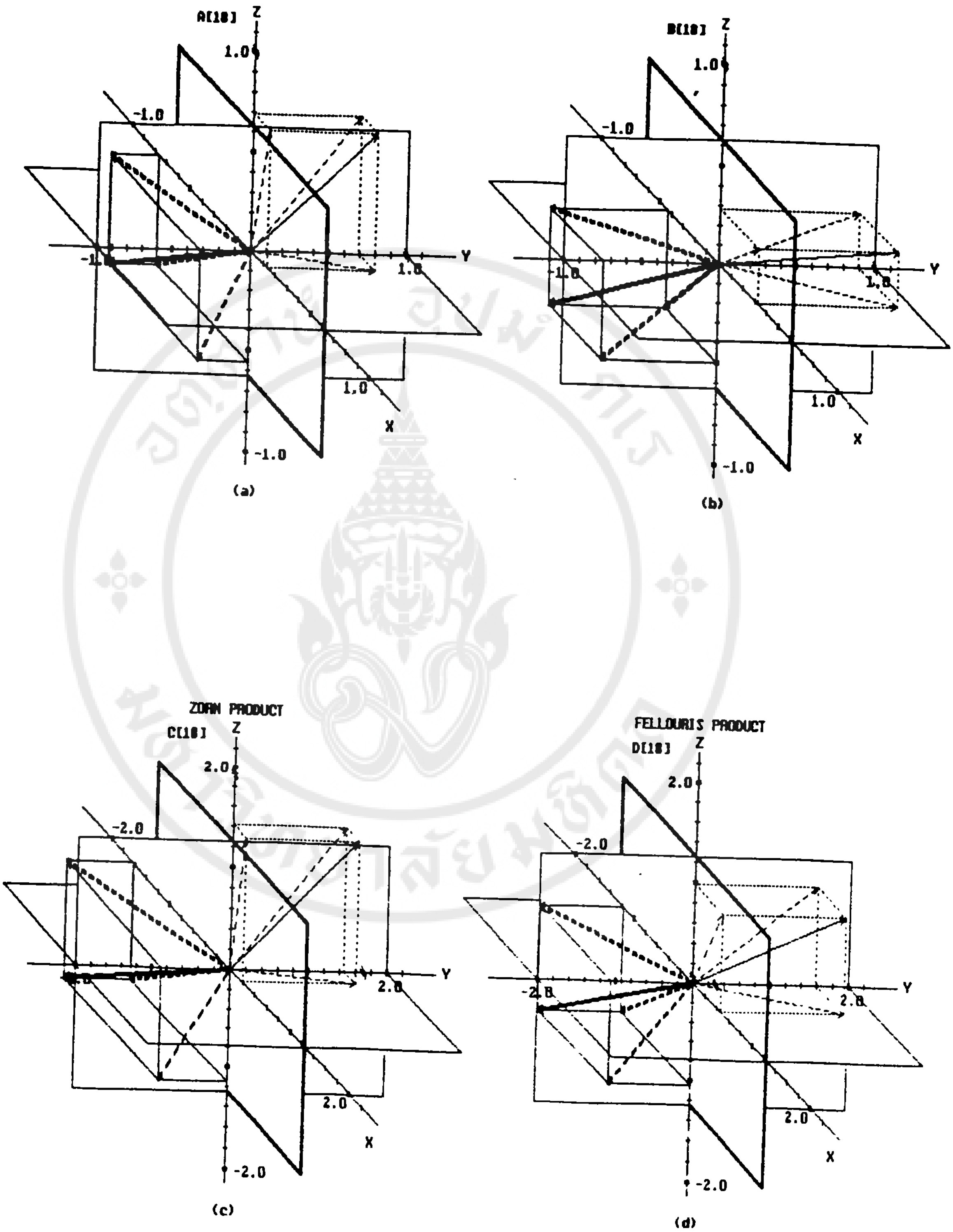


Fig. 4I(17) The representations of the vector matrices A , B , ZORN PRODUCT C , FELLOURIS PRODUCT D , MYUNG-ZORN PRODUCT E ($\tau = 0$) , and MYUNG-ZORN PRODUCT F ($\tau = \sqrt{a^2 + b^2 + |U^2| + |V^2|}$) , where A , B , C , D , E , and F $\in$ M ($M = \begin{bmatrix} a & U \\ V & b \end{bmatrix}$) , for data set 17. The scalars a and b are represented by asterisks on the Y and Z axes. The vectors U and V are represented by bold lines and ordinary lines, respectively.


A[18]
Z
1.0
-1.0
Y
-1.0
1.0
1.0
X
-1.0
(a)
B[18]
Z
1.0
-1.0
Y
1.0
X
-1.0
(b)
ZORN PRODUCT
C[18]
Z
2.0
-2.0
Y
2.0
2.0
X
-2.0
(c)
FELLOURIS PRODUCT
D[18]
Z
2.0
-2.0
Y
2.0
X
-2.0
(d)

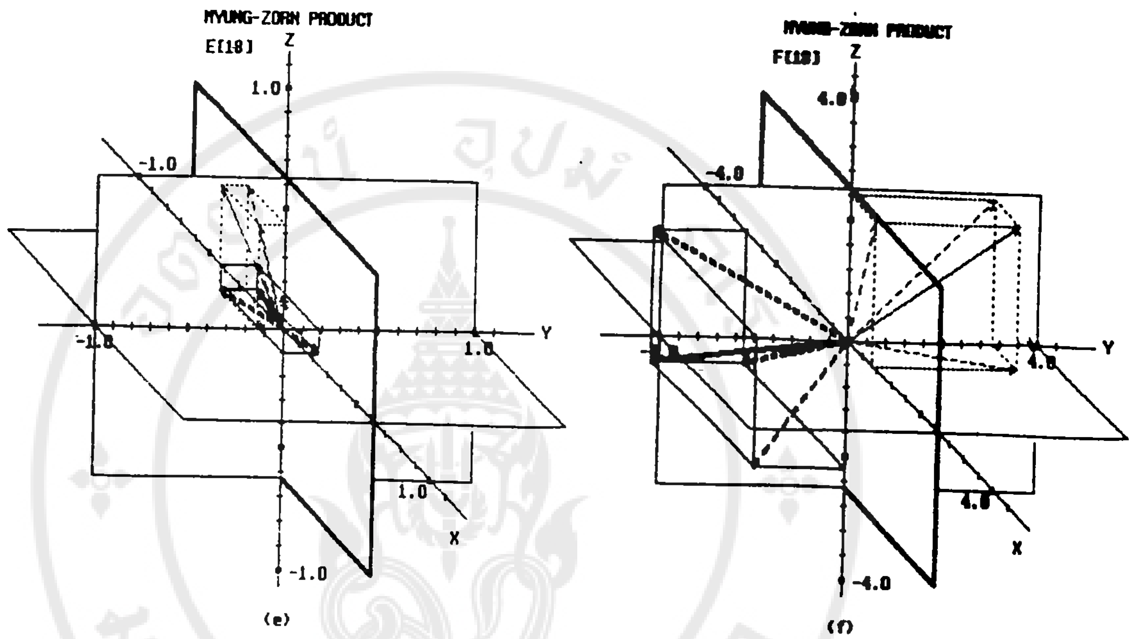
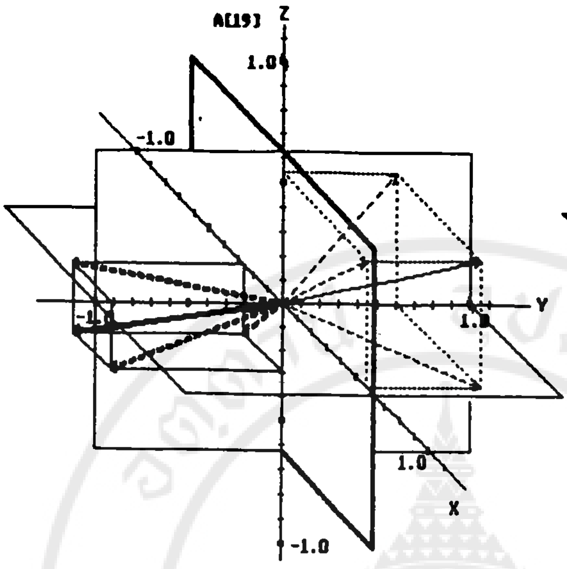
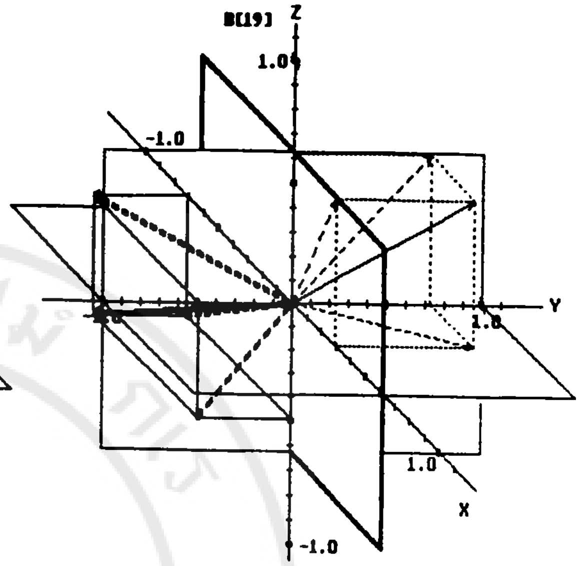


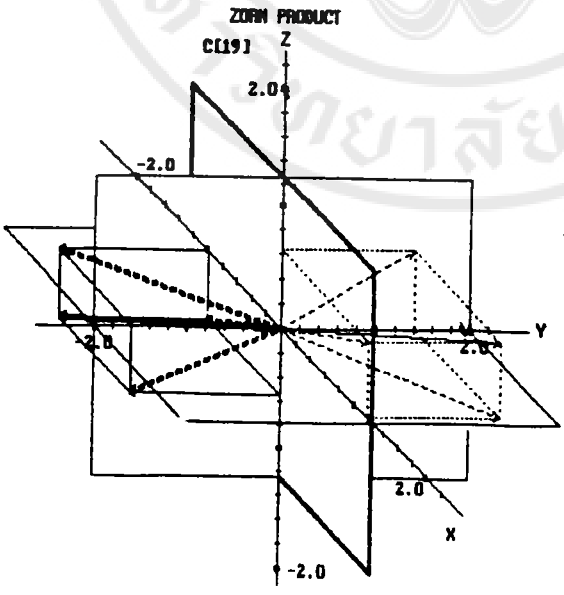
Fig. 4I(18) The representations of the vector matrices A , B , ZORN PRODUCT C , FELLOURIS PRODUCT D , MYUNG-ZORN PRODUCT E ($\tau = 0$) , and MYUNG-ZORN PRODUCT F ($\tau = \sqrt{a^2 + b^2 + |U^2| + |V^2|}$) , where A , B , C , D , E , and F $\in$ M (M = $\begin{bmatrix} a & U \\ V & b \end{bmatrix}$) , for data set 18. The scalars a and b are represented by asterisks on the Y and Z axes. The vectors U and V are represented by bold lines and ordinary lines, respectively.


A[19]
Z
1.0
-1.0
Y
1.0
X
-1.0

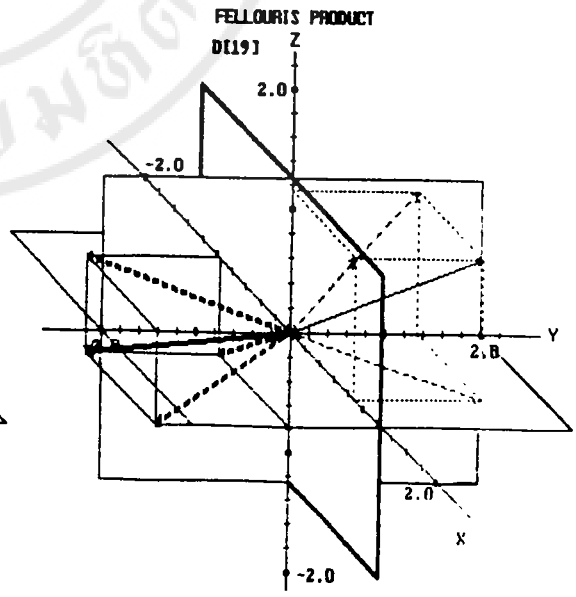
(a)


B[19]
Z
1.0
-1.0
Y
1.0
X
-1.0

(b)


ZORN PRODUCT
C[19]
Z
2.0
-2.0
Y
2.0
X
-2.0

(c)


FELLOURIS PRODUCT
D[19]
Z
2.0
-2.0
Y
2.0
X
-2.0

(d)

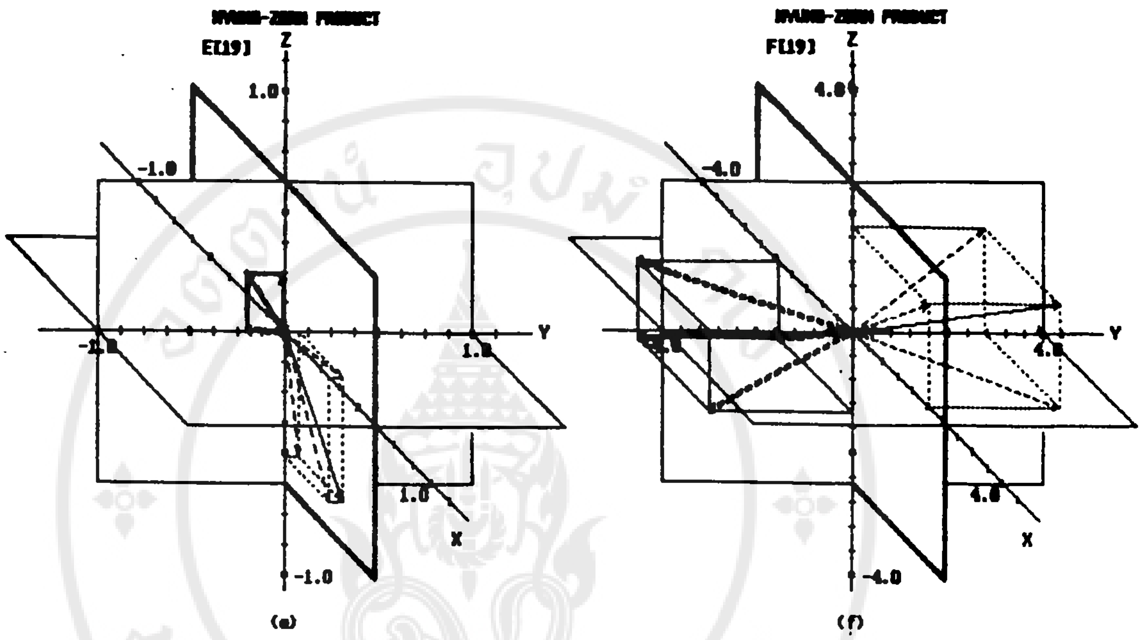
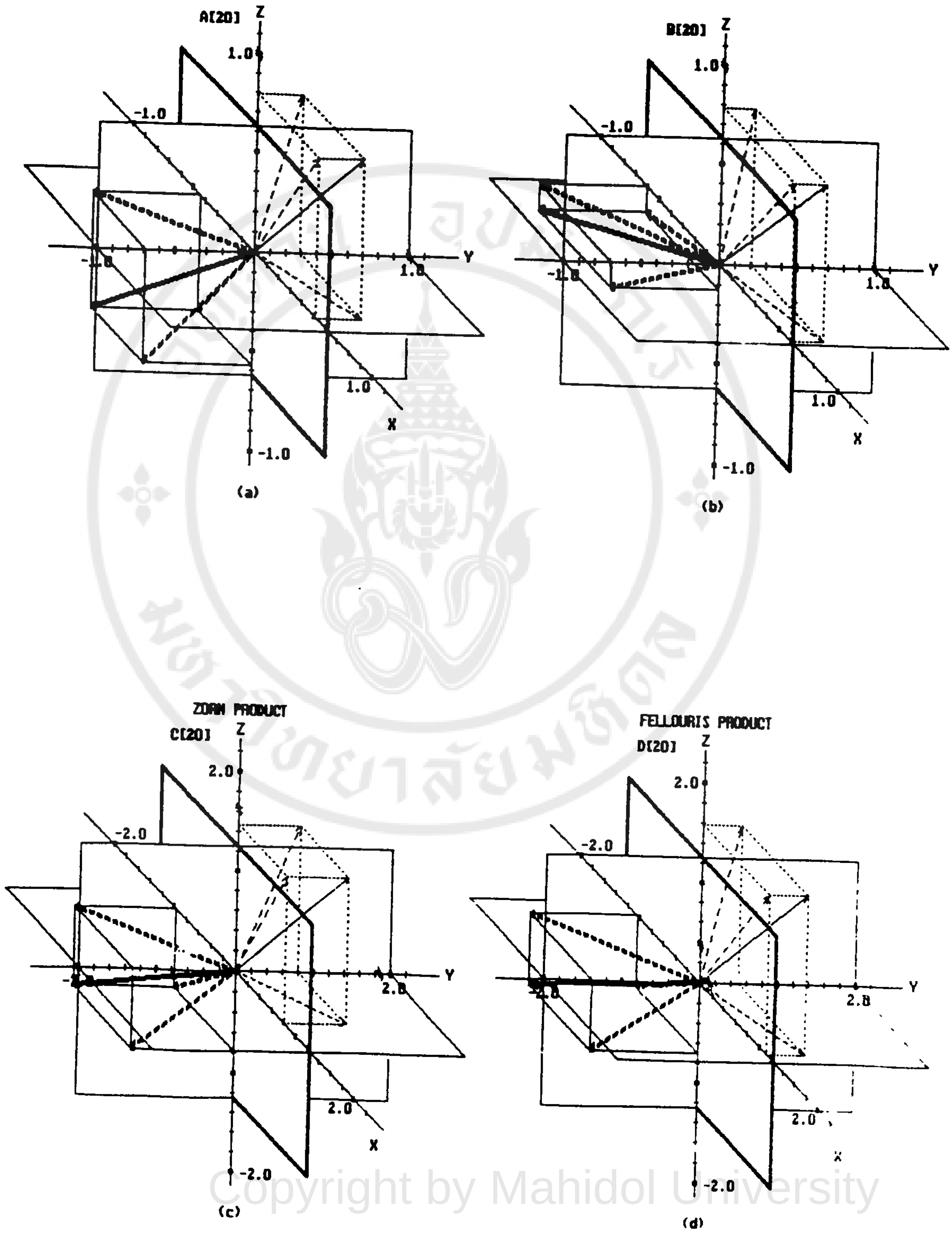


Fig. 4I(19) The representations of the vector matrices A , B , ZORN PRODUCT C , FELLOURIS PRODUCT D , MYUNG-ZORN PRODUCT E ($\tau = 0$) , and MYUNG-ZORN PRODUCT F ($\tau = \sqrt{a^2 + b^2 + |U^2| + |V^2|}$) , where A , B , C , D , E , and F $\in$ M ($M = \begin{bmatrix} a & U \\ V & b \end{bmatrix}$) , for data set 19. The scalars a and b are represented by asterisks on the Y and Z axes. The vectors U and V are represented by bold lines and ordinary lines, respectively.


A[20]
Z
1.0
-1.0
Y
1.0
-1.0
X
1.0
-1.0
(a)
B[20]
Z
1.0
-1.0
Y
1.0
-1.0
X
1.0
-1.0
(b)
ZORN PRODUCT
C[20]
Z
2.0
-2.0
Y
2.0
X
2.0
-2.0
(c)
FELLOURIS PRODUCT
D[20]
Z
2.0
-2.0
Y
2.0
-2.0
X
2.0
-2.0
(d)

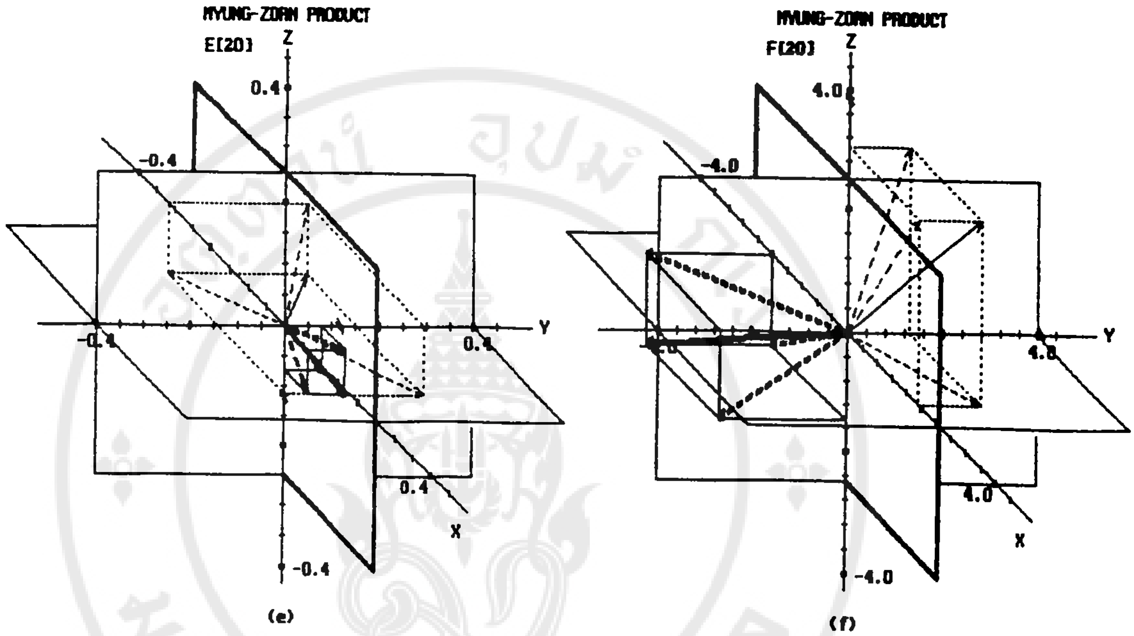


Fig. 41(20) The representations of the vector matrices A , B , ZORN PRODUCT C , FELLOURIS PRODUCT D , MYUNG-ZORN PRODUCT E ($\tau = 0$) , and MYUNG-ZORN PRODUCT F ($\tau = \sqrt{a^2 + b^2 + |U^2| + |V^2|}$) , where A , B , C , D , E , and F $\in$ M ($M = \begin{bmatrix} a & U \\ V & b \end{bmatrix}$) , for data set 20. The scalars a and b are represented by asterisks on the Y and Z axes. The vectors U and V are represented by bold lines and ordinary lines, respectively.

CHAPTER V

APPLICATIONS OF GENERALIZED VECTOR MATRICES, HYPERCOMPLEX NUMBERS, AND DUAL NUMBERS

"In every mathematical investigation, the question will arise whether we can apply our mathematical results to the real world."

[Vladimir Igorevich ARNOL'D (1937-)]

In this chapter we will consider the applications of generalized vector matrices and certain types of hypernumbers, viz. hypercomplex and dual numbers, to mechanics, robotics, and optics. Section 5.1 gives the applications of dual numbers and quaternions, and, in particular, the dual quaternions (quaternions with dual number components), to the analysis of multi-rigid-body and gyroscopic systems. Applications of generalized vector matrices and of hypercomplex and dual numbers to robotics and optics are given, respectively, in Sections 5.2 and 5.3.

Section 5.1 Application to Mechanics : Multi-Rigid-Body and Gyroscopic Systems

In recent years, there has been considerable interest in the application of dual numbers and quaternions to the kinematic analysis of spatial mechanisms. The pioneering work in using dual numbers in applied

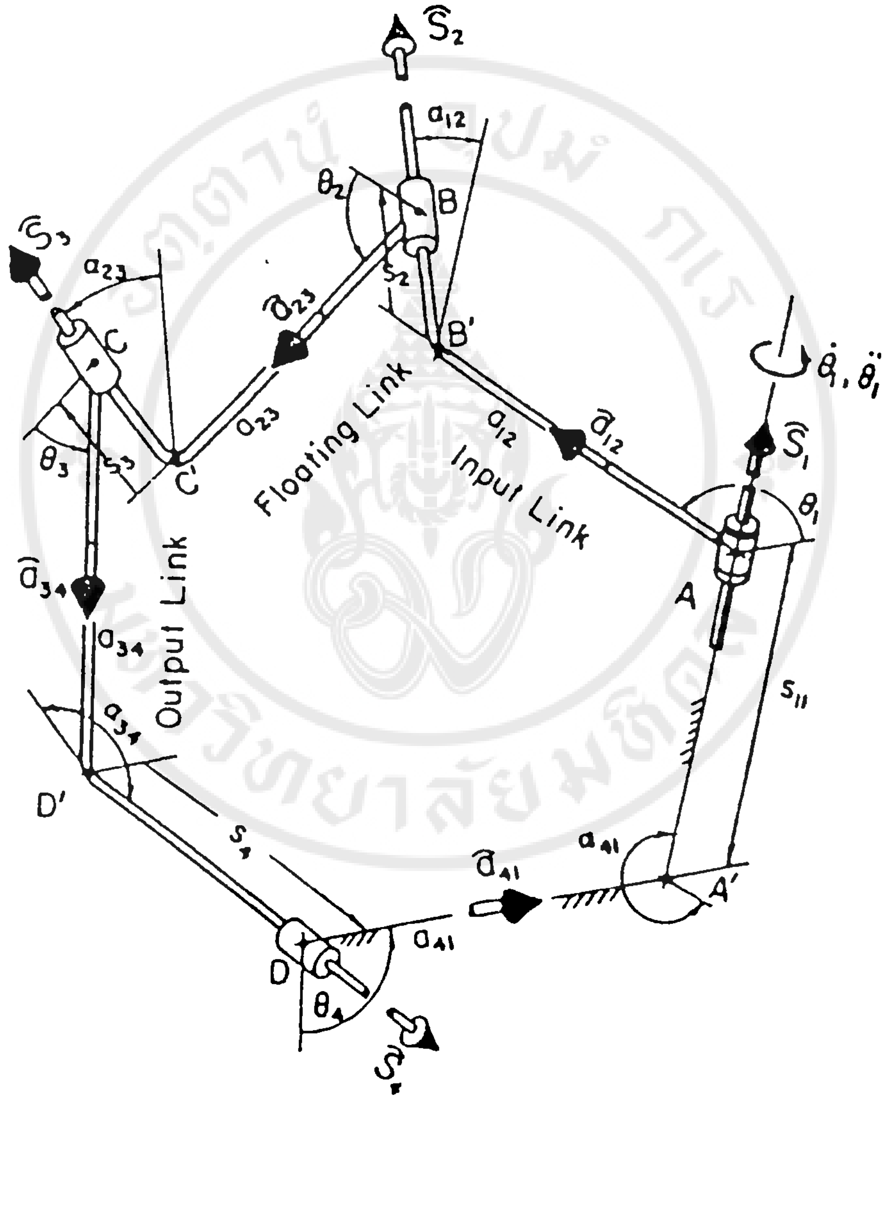


Fig. 5.1A A RCCC mechanism

mechanics was developed by Dimentberg (1965), Denavit (1958), and Hartenberg. Dimentberg applied dual numbers particularly in screw calculus, and Denavit and Hartenberg combined dual numbers with matrix coordinate-transformations for displacement analysis of several basic spatial mechanisms, including the RCCC mechanism (one revolute and three cylindric pairs), as shown in Fig 5.1A .

The RCCC mechanism is capable of transforming a rotary motion into a variable-pitch helical motion. The motions of its input link and output link are simple : The input link rotates about the input axis, the output link rotates about and slides along the output axis, both axes being fixed in space. The motion of the floating link, however, is relatively complex : The link spins about and slides along one floating axis attached to the input link which precesses about the input axis and, at the same time, the floating link spins about and slides along another floating axis attached to the output link which rotates about and slides along the output axis.

In general, we may describe the motion of the floating link as a combination of rotation and sliding about and along an instantaneous screw axis which involves location change as well as changes in direction relative to the fixed link. The floating link, therefore, is subject to both Coriolis and gyroscopic effects. ([20],[31],[36],[41],[46],[61],[71],[76],[78],[79],[80],[81],[82],[84],[85]) 5.1.1 Rotating coordinate systems

The following is a brief survey of the relation between a rotating coordinate system and a fixed inertial coordinate frame, and includes a discussion of the relationship between a moving coordinate system (rotating and translating) and an inertial frame. In Fig.5.1B, two right handed coordinate systems are given : an unprimed coordinate system OXYZ (inertial frame) and a primed coordinate system $O'X'Y'Z'$(rotating frame), whose origins are coincident at a point O, and the axes OX',OY', and OZ' are rotating relative to the axes OX, OY, and OZ, respectively. Let (**i**,**j**,**k**) and ($\mathbf{i}'$,$\mathbf{j}'$,$\mathbf{k}'$) be their respective unit vectors along the principal axes.

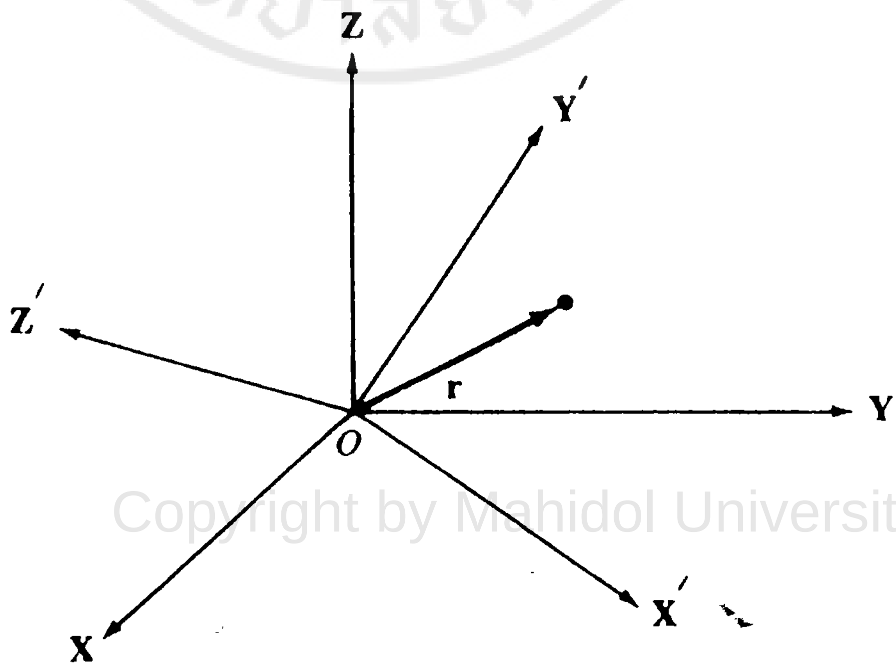


Fig. 5.1B The rotating coordinate system

A vector **r** can be expressed as

$$\mathbf{r} = x\mathbf{i} + y\mathbf{j} + z\mathbf{k} \quad , \qquad \text{----- (5.1.1)}$$

$$\mathbf{r} = x'\mathbf{i}' + y'\mathbf{j}' + z'\mathbf{k}' \quad . \qquad \text{----- (5.1.2)}$$

The two time derivatives are :

$\frac{d\mathbf{r}}{dt}$, the time derivative with respect to the fixed reference coordinate system,

and $\frac{d\mathbf{r}}{dt'}$, the time derivative with respect to the primed coordinate system which is rotating.

Differentiating Eq.(5.1.2) with respect to t′ and t, respectively, we get

$$\frac{d\mathbf{r}}{dt'} = \frac{dx'}{dt'}\mathbf{i}' + \frac{dy'}{dt'}\mathbf{j}' + \frac{dz'}{dt'}\mathbf{k}'$$

$$= \dot{x}'\mathbf{i}' + \dot{y}'\mathbf{j}' + \dot{z}'\mathbf{k}' \quad , \qquad \text{----- (5.1.3)}$$

$$\frac{d\mathbf{r}}{dt} = \frac{dx'}{dt}\mathbf{i}' + \frac{dy'}{dt}\mathbf{j}' + \frac{dz'}{dt}\mathbf{k}' + x'\frac{d\mathbf{i}'}{dt} + y'\frac{d\mathbf{j}'}{dt} + z'\frac{d\mathbf{k}'}{dt} \quad . \qquad \text{----- (5.1.4)}$$

From Eqs.(5.1.3) and (5.1.4), we get

$$\frac{d\mathbf{r}}{dt} = \frac{d\mathbf{r}}{dt'} + x'\frac{d\mathbf{i}'}{dt} + y'\frac{d\mathbf{j}'}{dt} + z'\frac{d\mathbf{k}'}{dt} \quad . \qquad \text{----- (5.1.5)}$$

In order to find a relationship between the primed and unprimed derivatives, let us suppose that the primed coordinate system is rotating about some axis OQ passing through the origin O, with angular velocity **ω** (see Fig.5.1C), then the angular velocity **ω** is defined as a vector of magnitude ω directed along the axis OQ in the direction of a right-handed rotation with the primed coordinate system.

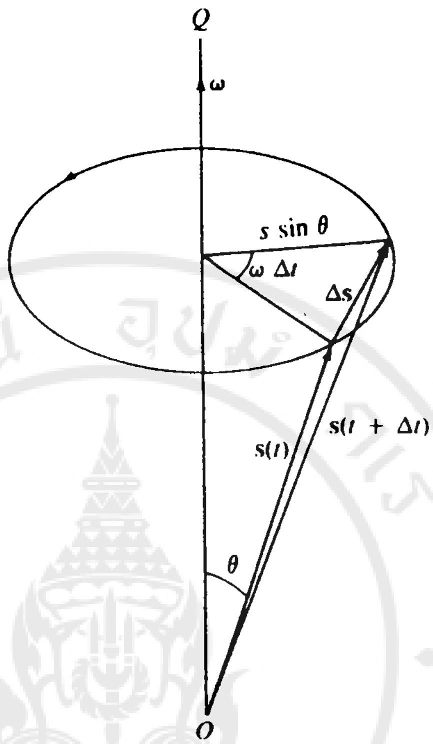


Fig. 5.1C Time derivative of a rotating coordinate system.

Consider a vector s(t) of constant magnitude rotating about OQ as in Fig.5.1C. In an infinitesimal time interval δt the vector s will have changed by

$$\delta s = |s|\sin\theta|\omega|\delta t \qquad \text{----- (5.1.6)}$$

in the direction perpendicular to both s and ω. That is,

$$\delta s = (\omega \times s)\delta t \quad . \qquad \text{----- (5.1.7)}$$

Hence $$\frac{ds}{dt} = \lim_{\delta t \to 0} \frac{\delta s}{\delta t} = \omega \times s \quad . \qquad \text{----- (5.1.8)}$$

On applying Eq.(5.1.8) to the unit vector $(\mathbf{i}', \mathbf{j}', \mathbf{k}')$, then Eq.(5.1.5) becomes

$$\frac{d\mathbf{r}}{dt} = \frac{d\mathbf{r}}{dt'} + x'(\boldsymbol{\omega}\times\mathbf{i}') + y'(\boldsymbol{\omega}\times\mathbf{j}') + z'(\boldsymbol{\omega}\times\mathbf{k}') \quad \text{-----} \quad (5.1.9)$$

$$= \frac{d\mathbf{r}}{dt'} + \boldsymbol{\omega}\times\mathbf{r} \quad . \quad \text{-----} \quad (5.1.10)$$

Furthermore, we have

$$\frac{d^2\mathbf{r}}{dt^2} = \frac{d}{dt}\left[\frac{d\mathbf{r}}{dt'}\right] + \boldsymbol{\omega}\times\frac{d\mathbf{r}}{dt} + \frac{d\boldsymbol{\omega}}{dt}\times\mathbf{r} \quad \text{-----} \quad (5.1.11)$$

$$= \frac{d^2\mathbf{r}}{dt'^2} + \boldsymbol{\omega}\times\frac{d\mathbf{r}}{dt'} + \boldsymbol{\omega}\times\left[\frac{d\mathbf{r}}{dt'} + \boldsymbol{\omega}\times\mathbf{r}\right] + \frac{d\boldsymbol{\omega}}{dt}\times\mathbf{r}$$

$$= \frac{d^2\mathbf{r}}{dt'^2} + 2\boldsymbol{\omega}\times\frac{d\mathbf{r}}{dt'} + \boldsymbol{\omega}\times(\boldsymbol{\omega}\times\mathbf{r}) + \frac{d\boldsymbol{\omega}}{dt}\times\mathbf{r} \ . \quad \text{-----} \quad (5.1.12)$$

Eq.(5.1.12) is called the *Coriolis theorem*. The meanings of various terms in the above equation are as follows :

$\frac{d^2\mathbf{r}}{dt'^2}$ is the acceleration relative to the primed coordinate system,

$2\boldsymbol{\omega}\times\frac{d\mathbf{r}}{dt'}$ is the Coriolis acceleration,

$\boldsymbol{\omega}\times(\boldsymbol{\omega}\times\mathbf{r})$ is the centripetal acceleration.

Note that the term $\frac{d\boldsymbol{\omega}}{dt}\times\mathbf{r}$ is zero for constant angular velocity of rotation about a fixed axis.

5.1.2 Moving coordinate systems

A useful extension of the above rotating coordinate systems is to include a translational motion of the primed coordinate system with respect to the unprimed coordinate system. From Fig.5.1D, the primed coordinate system $O'X'Y'Z'$ is rotating and translating with respect to the unprimed coordinate system OXYZ which is an inertial frame. A particle P with mass M is located by vectors $\mathbf{r}$ and $\mathbf{r}'$ with respect to the origins of the coordinate frames OXYZ and $O'X'Y'Z'$, respectively. The origin O' is located by a vector $\mathbf{h}$ with respect to the origin O. The relation between the position vectors $\mathbf{r}$ and $\mathbf{r}'$ is given by (see Fig.5.1D)

$$\mathbf{r} = \mathbf{r}' + \mathbf{h} \qquad \text{-----} \ (5.1.13)$$

If the primed coordinate system $O'X'Y'Z'$ is moving (rotating and translating) with respect to the unprimed coordinate system OXYZ, then

$$\mathbf{V}(t) = \frac{d\mathbf{r}}{dt} = \frac{d\mathbf{r}'}{dt} + \frac{d\mathbf{h}}{dt} ,$$

$$\mathbf{V} = \mathbf{V}' + \mathbf{V}_h , \qquad \text{-----} \ (5.1.14)$$

where $\mathbf{V}'$ and $\mathbf{V}$ are the velocities of the moving particle P relative to the coordinate frames $O'X'Y'Z'$ and OXYZ, respectively, and $\mathbf{V}_h$ is the velocity of the primed coordinate system $O'X'Y'Z'$ relative to the unprimed coordinate system OXYZ. On using Eq.(5.1.10), Eq.(5.1.14) can be expressed as

$$V(t) = \frac{dr'}{dt'} + \omega \times r' + \frac{dh}{dt} \quad . \qquad \text{----- (5.1.15)}$$

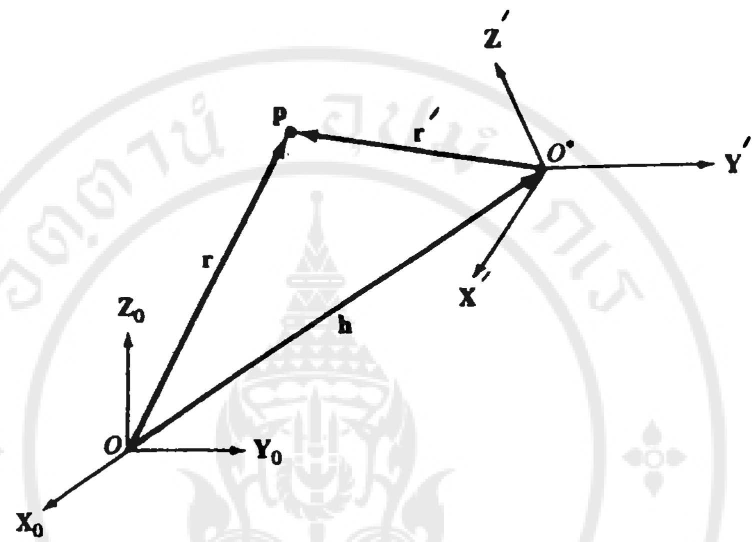


Fig. 5.1D Moving coordinate system

Similarly, the acceleration of the particle P with respect to the unprimed coordinate system is

$$a(t) = \frac{dV(t)}{dt} = \frac{d^2r'}{dt^2} + \frac{d^2h}{dt^2} \quad ,$$

or

$$a(t) = a' + a_h \quad . \qquad \text{----- (5.1.16)}$$

That is,

$$a(t) = \frac{d^2r'}{dt'^2} + \frac{2\omega \times dr'}{dt'} + \omega \times (\omega \times r') + \frac{d\omega \times r'}{dt} + \frac{d^2h}{dt^2} \quad . \qquad \text{----- (5.1.17)}$$

In the following sections, the above concepts will be applied to the kinematics of the links (see Fig.5.1A), and the corresponding developments using dual numbers will be considered.

5.1.3 Dual numbers

Dual numbers play an important role in the dynamics of multi-rigid-body open chain systems. The technique using dual numbers has been successfully applied to the study of the dynamics of generalized gyroscopic systems. Now, we will introduce the *dual number* $\hat{A} = A + \varepsilon A_o$, where A is its *primary part*, A_o its *dual part*, and ε is an operator (roughly analogous to the imaginary i treated as an operator in a complex number) such that

$$\varepsilon \neq 0 , \quad \varepsilon^2 = \varepsilon^3 = \varepsilon^4 = \ldots = 0 . \qquad \text{-----} \ (5.1.18)$$

We can say that the *dual symbol* ε is a divisor of zero and is nilpotent. It can also be regarded as a special parameter ε in expressions involving dual numbers. From Chapter III, Section 3, the parameter ε can be found from the formula

$$\varepsilon = k(i_n \pm \varepsilon_n) , \qquad \text{-----} \ (5.1.19)$$

where k is a real finite number, i_n and ε_n are hypernumbers, and $n \neq m$. For instance, $\varepsilon = k(i_1 + \epsilon_2)$, therefore,

$$\varepsilon = \begin{bmatrix} 1 & -1 \\ 1 & -1 \end{bmatrix} \quad \text{and} \quad \varepsilon^2 = 0 .$$

Since the dual parameter ε can be derived from Eq.(5.2.19), and depends on the hypernumbers i_n and ε_n, it can also be regarded as a hypernumber.

5.1.4 Unit line vectors and representations of reference frames

A unit line vector (ULV) is a unit vector bound to a definite line. For example, in Fig.5.1E, a unit vector **n** is restricted to lie on a line N .

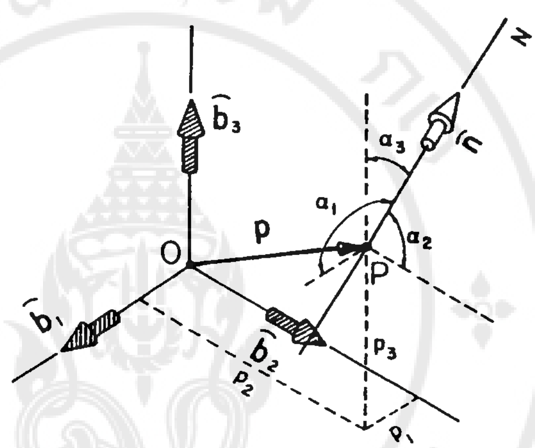


Fig. 5.1E Specification of a line in a reference frame

The ULV can be specified by the dual vector :

$$\hat{\mathbf{n}} = \mathbf{n} + \varepsilon \mathbf{n}_o \quad . \qquad ----- (5.1.20)$$

The primary part **n** is a unit vector parallel to line N ; the dual part $\mathbf{n}_o = \mathbf{p} \times \mathbf{n}$ (**p** is the position vector of an arbitrarily chosen point P on line n) prescribes the position of line N relative to origin O. The dual vector may be written as

$$\hat{n}_i = n_i + \varepsilon \epsilon_{i\lambda\nu} p_\lambda n_\nu \quad , \qquad ----- (5.1.21)$$

where $\epsilon_{i\lambda\nu}$ is the Levi-Civita symbol, $n_i = \cos \alpha_i$ are direction cosines of **n**, and P_λ are the coordinates of point P on line N. Thus $\hat{n}_i$ represents the cosine of the dual angle between n and $\hat{b}_i$.

5.1.5 Dual-transformation matrix

Consider two arbitrarily chosen right-handed Cartesian reference frames $\{\hat{V}\}_B$ and $\{\hat{U}\}_A$ as shown in Fig.5.1F. The origins of the two reference frames are, in general, noncoincident. A transformation of the components of a given dual vector specified in $\{\hat{V}\}_B$ to those in $\{\hat{U}\}_A$ is completely described by a 3×3 matrix with dual-number elements, referred to as *dual-transformation matrix*. In accordance with Eq.(5.1.21), we may write the nine dual elements as

$$\hat{b}_{ij} = b_{ij} + \varepsilon\epsilon_{j\alpha\beta} a_\alpha b_{i\beta} \quad . \qquad \text{----- (5.1.22)}$$

As the $\hat{U}_i$ are mutually orthogonally intersecting, we may write the relation

$$\hat{b}_{i\rho}\hat{b}_{k\rho} = \delta_{ik} = \begin{cases} 1 & \text{for } i = k \ , \\ 0 & \text{for } i \neq k \ . \end{cases} \qquad \text{----- (5.1.23)}$$

Eq.(5.1.23) indicates that the dual-transformation matrix retains its orthogonality properties, and hence, its inverse is identical to its transposed matrix. Eq.(5.1.22) consists of six dual scalar equations. The restrictions imply that among the nine dual elements, only three are independent. Eq.(5.1.23) confirms that six independent real parameters are neccessary and sufficient to specify the relative position between any two given reference frames whose origins are not coincident.

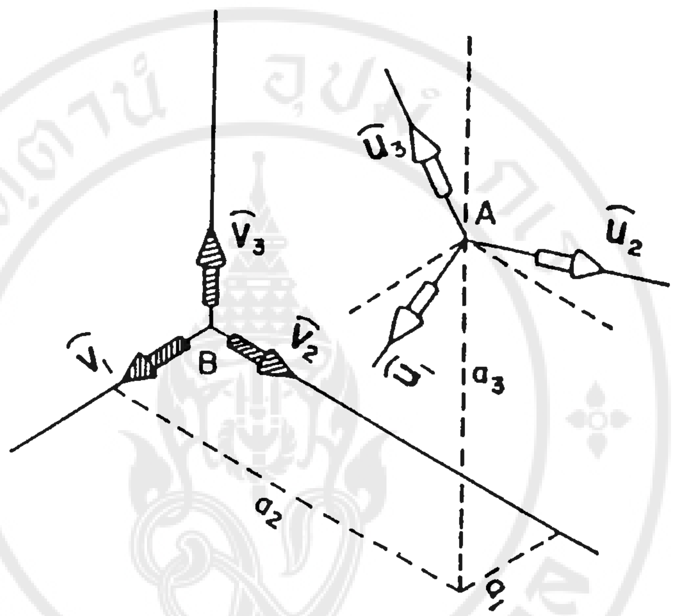


Fig. 5.1F Two reference frames in space

From Eq.(5.1.23), if $a_1 = a_2 = a_3 = 0$, then the dual part vanishes. This implies that A coincides with B (see Fig.5.1F). A dual vector is the mixture of two independent vectors. The 3×3 dual-transformation matrix, with the direction cosines as its elements, is formed by nine position numbers, as given in Eq.(5.1.22). For the two arbitrarily chosen reference frames as shown in Fig.5.1F, we may express the transformation for a given dual vector whose components are V_β referred to $\{\hat{V}\}_B$ and $\hat{U}_i$ referred to $\{\hat{U}\}_A$ as

$$\hat{U}_i = b_{i\beta} V_\beta \quad , \qquad \text{-----} \; (5.1.24)$$

where b_{ip} are elements of the dual-transformation matrix as given in Eq.(5.1.22). The inverse transformation is

$$\hat{V}_i = \hat{b}_{pi}\hat{U}_p \quad . \qquad ----- (5.1.25)$$

5.1.6 Spin axis, tilt axis, and azimuth axis of the offset unsymmetric gyroscope

An *offset unsymmetric* gyroscope with *oblique* rotor refers to the system which consists of a heavy rigid asymmetric body with mass center E located away from the spin axis, as shown in Fig. 5.1H. In general, none of its principal central axes (e_1,e_2,e_3) is parallel to the spin axis. The rotor is suspended in inertial space by two gimbals of negligible mass. Consider Fig.5.1H ; the definition of the independent variables which are employed in the kinematic description of the offset unsymmetric gyroscope with oblique rotor is shown in Table 5.1I.

Table 5.1I Reference frames

symbol	Definition	Attached to
$\{e\}_E$	principal central frame (PCF)	rotor
$\{c\}_C$	rotor geometric frame (RGF)	rotor
$\{b\}_B$	inner gimbal frame (IGF)	inner gimbal
$\{a\}_A$	outer gimbal frame (OGF)	outer gimbal
$\{i\}_O$	inertial reference frame (IRF)	inertial space

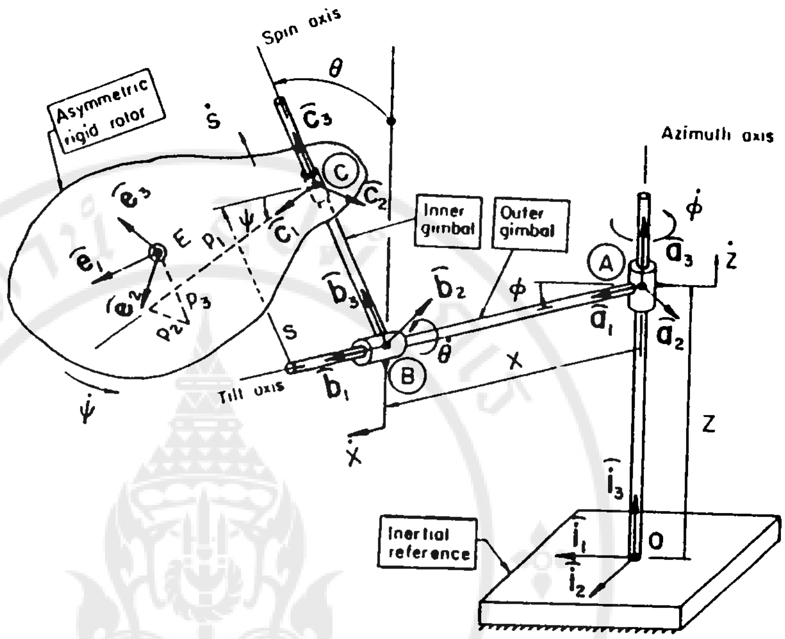


Fig. 5.1H An offset unsymmetric gyroscope with oblique rotor

From Fig.5.1H, we find that $\mathbf{c}_3$ coincides with $\mathbf{b}_3$ (axis of pair C), $\mathbf{b}_1$ coincides with $\mathbf{a}_1$ (axis of pair B), and $\mathbf{a}_3$ coincides with $\mathbf{i}_3$ (axis of pair A). The three axes of cylindric pairs C, B, and A are referred to, respectively, as the *spin axis*, *tilt axis*, and *azimuth axis*. The azimuth axis is fixed in inertial space but the other two, in the most general case, change direction as well as location during motion. The spin, tilt and azimuth axes are pertinent to visualization of the kinematic relation among various members of the offset unsymmetric gyroscope with oblique rotor in inertial space.

5.1.7 Dual Eulerian angles

The six independent rotation and sliding variables ϕ, θ, ψ, z, x, and s (see Fig.5.1G) may be interpreted as a set of dual Eulerian angles as follows :

$$\hat{\phi} = \phi + \varepsilon z \,,$$
$$\hat{\theta} = \theta + \varepsilon x \,, \qquad \text{----- (5.1.26)}$$
$$\hat{\psi} = \psi + \varepsilon s \,.$$

Using Eq.(5.1.22), we may express the three screw displacements as

$$[\hat{\phi}]_3 = \begin{bmatrix} C\hat{\phi} & S\hat{\phi} & 0 \\ -S\hat{\phi} & C\hat{\phi} & 0 \\ 0 & 0 & 1 \end{bmatrix},$$

$$[\hat{\theta}]_1 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & C\hat{\theta} & S\hat{\theta} \\ 0 & -S\hat{\theta} & C\hat{\theta} \end{bmatrix}, \qquad \text{----- (5.1.27)}$$

$$[\hat{\psi}]_3 = \begin{bmatrix} C\hat{\psi} & S\hat{\psi} & 0 \\ -S\hat{\psi} & C\hat{\psi} & 0 \\ 0 & 0 & 1 \end{bmatrix},$$

where $C\hat{\theta}$ and $S\hat{\theta}$ are cosine $\hat{\theta}$ and sine $\hat{\theta}$. The resultant transformation matrix of $\{c\}_c$ with respect to $\{a\}_A$ may be expressed as the product of the three screw matrices

$$[\hat{b}] = [\hat{\psi}]_3[\hat{\theta}]_1[\hat{\phi}]_3 \quad ,$$

i.e.,
$$[\hat{b}] = \begin{bmatrix} C\hat{\psi}C\hat{\phi}-S\hat{\psi}C\hat{\theta}S\hat{\phi} & C\hat{\psi}S\hat{\theta}+S\hat{\psi}C\hat{\theta}C\hat{\phi} & S\hat{\psi}S\hat{\theta} \\ -S\hat{\psi}C\hat{\phi}-C\hat{\psi}C\hat{\theta}S\hat{\phi} & -S\hat{\psi}S\hat{\phi}+C\hat{\psi}C\hat{\theta}C\hat{\phi} & C\hat{\psi}S\hat{\theta} \\ S\hat{\theta}S\hat{\phi} & -S\hat{\theta}C\hat{\phi} & C\hat{\theta} \end{bmatrix} .$$

----- (5.1.28)

The three dual Eulerian angles $\hat{\phi}$ (dual azimuth angle), $\hat{\theta}$ (dual nutation angle), and $\hat{\psi}$ (dual rotor angle) completely specify the position of RGF in IRF (see Table 5.1H).

5.1.8 Dual velocity components referred to the rotor geometric frame (RGF)

In the general configuration of the system, as shown in Fig.5.1H, the dual Eulerian angles are time-dependent. The time derivatives of these angles are

$\dot{\hat{\phi}} = \dot{\phi} + \varepsilon\dot{z}$ = the dual velocity of the outer gimbal about i_3 axis : referred to as the *dual precession rate* ($\dot{\phi}$ is the precession rate and $\dot{z}$ is the sliding velocity along the azimuth axis).

$\dot{\hat{\theta}} = \dot{\theta} + \varepsilon\dot{x}$ = the dual angle between the fixed axis i_3 and the spin axis c_3, having a positive sense with respect to b_1 : referred to as the *dual nutation rate* ($\dot{\theta}$ is the nutation rate and $\dot{x}$ is the sliding velocity along the tilt axis).

$\dot{\hat{\psi}} = \dot{\psi} + \varepsilon\dot{s}$ = the dual angle between the line of nodes b_1 and the body axis c_1, having a positive sense with respect to c_3 : referred to as the *dual spin velocity* ($\dot{\psi}$ is the spin velocity and $\dot{s}$ is the sliding velocity along the spin axis).

Now, the components of the dual velocity of the rotor referred to the rotor *rigid frame* (RGF), $\{c\}_c$, are denoted by

$$\hat{V}_1 = \omega_1 + \varepsilon V_1 \quad . \qquad \text{----- (5.1.29)}$$

The relationship between the two moving reference frames $\{a\}_A$ and $\{c\}_c$ can be specified through the use of screw operators as

$$a_1 = b_1 = c_1 C\hat{\psi} - c_2 S\hat{\psi} \quad , \qquad \text{----- (5.1.30)}$$

$$a_3 = i_3 = c_3 C\hat{\theta} + b_2 S\hat{\theta} \quad , \qquad \text{----- (5.1.31)}$$

$$b_2 = b_3 \times b_1 = c_3 \times (c_1 C\hat{\psi} - c_2 S\hat{\psi}) \quad .$$

Substituting b_2 into Eq.(5.1.31), we get

$$a_3 = i_3 = c_1 S\hat{\theta} S\hat{\psi} + c_2 S\hat{\theta} C\hat{\psi} + c_3 C\hat{\theta} \quad , \qquad \text{----- (5.1.32)}$$

$$a_2 = a_3 \times a_1 = \begin{bmatrix} c_1 & c_2 & c_3 \\ S\hat{\theta}S\hat{\psi} & S\hat{\theta}C\hat{\psi} & C\hat{\theta} \\ C\hat{\psi} & -S\hat{\psi} & 0 \end{bmatrix} ,$$

$$a_2 = c_1 (C\hat{\theta} S\hat{\psi}) - c_2 (-C\hat{\theta} C\hat{\psi}) + c_3 (-S\hat{\theta} S^2\hat{\psi} - S\hat{\theta} C^2\hat{\psi}) \quad .$$

Therefore ,

$$a_2 = c_1 C\hat{\theta} S\hat{\psi} + c_2 C\hat{\theta} C\hat{\psi} - c_3 S\hat{\theta} \quad . \qquad \text{----- (5.1.33)}$$

The dual velocity of the asymmetric top may be specified as

$$\begin{aligned}\hat{V} &= \dot{\hat{\phi}} i_3 + \dot{\hat{\theta}} b_1 + \dot{\hat{\psi}} c_3 \\ &= \dot{\hat{\phi}}(c_1 S\hat{\theta}S\hat{\psi} + c_2 S\hat{\theta}C\hat{\psi} + c_3 C\hat{\theta}) + \dot{\hat{\theta}}(c_1 C\hat{\psi} - c_2 S\hat{\psi}) + \dot{\hat{\psi}} c_3 \\ &= c_1(\dot{\hat{\phi}}S\hat{\theta}S\hat{\psi} + \dot{\hat{\theta}}C\hat{\psi}) + c_2(\dot{\hat{\phi}}S\hat{\theta}C\hat{\psi} - \dot{\hat{\theta}}S\hat{\psi}) + c_3(\dot{\hat{\phi}}C\hat{\theta} + \dot{\hat{\psi}}) \quad .\end{aligned}$$

----- (5.1.34)

On using the condition of

$$\begin{aligned}\sin(\alpha+\varepsilon\beta) &= \sin(\alpha)\cos(\varepsilon\beta) + \cos(\alpha)\sin(\varepsilon\beta) \\ &= \sin(\alpha) + \varepsilon\beta\cos(\alpha) \quad , \\ \cos(\alpha+\varepsilon\beta) &= \cos(\alpha) - \varepsilon\beta\sin(\alpha) \quad ,\end{aligned}$$

where $\sin(\varepsilon\beta) = \varepsilon\beta$ and $\cos(\varepsilon\beta) = 1$, in Eq.(5.1.34), the primary parts of Eq.(5.1.34) are given by

$$\begin{aligned}\omega_1 &= \dot{\phi}S\theta S\psi + \dot{\theta}C\psi \quad , \\ \omega_2 &= \dot{\phi}S\theta C\psi - \dot{\theta}S\psi \quad , \\ \omega_3 &= \dot{\phi}C\theta + \dot{\psi} \quad .\end{aligned} \qquad \text{----- (5.1.35)}$$

The dual parts of Eq.(5.1.35), i.e., the linear velocity components, are

$$\left.\begin{aligned}V_1 &= \dot{z}S\theta S\psi + x\dot{\phi}C\theta S\psi + s\dot{\phi}S\theta C\psi + \dot{x}C\psi - s\dot{\theta}S\psi \quad , \\ V_2 &= \dot{z}S\theta C\psi + x\dot{\phi}C\theta C\psi - s\dot{\phi}S\theta S\psi - \dot{x}S\psi - s\dot{\theta}C\psi \quad , \\ V_3 &= \dot{z}C\theta - x\dot{\phi}S\theta + \dot{s} \quad .\end{aligned}\right\} \text{(5.1.36)}$$

5.1.9 General equations of motion

Let us, again, consider Fig.5.1H in which A, B, and C are cylindrical pairs. The rotor has mass M and principal moments of inertia I_1, I_2, and I_3, correspond, respectively, to the principal central axes e_1, e_2, and e_3. Referred to $\{c\}_c$, the coordinates of the mass center E are P_1, P_2, and P_3; the orientation of $\{e\}_E$ is prescribed by the nine direction cosines b_{ij}. Hence, elements $\hat{b}_{ij}$ of the 3×3 dual-transformation matrix may be computed by using Eq.(5.1.22). Both $\{e\}_E$ and $\{c\}_c$ are attached to the rotor; therefore, $\hat{b}_{ij}$ are time independent. Let

$$\hat{U}_i = \Omega_i + \varepsilon U_i \qquad \text{-----} \ (5.1.37)$$

be the dual velocity of the rotor referred to PCF $\{e\}_E$. Obtaining Eqs.(5.1.22) and (5.1.29) in Eq.(5.1.24), we get

$$\hat{U}_i = \hat{b}_{i\rho}\hat{V}_\rho = (b_{i\rho} + \varepsilon\epsilon_{\rho\alpha\beta}P_\alpha b_{i\beta})(\omega_\rho + \varepsilon V_\rho) \quad ,$$

$$\Omega_i + \varepsilon U_i = b_{i\rho}\omega_\rho + \varepsilon(b_{i\rho}V_\rho + \epsilon_{\rho\alpha\beta}P_\alpha b_{i\rho}\omega_\rho) \quad .$$

Therefore ,

$$\Omega_i = b_{i\rho}\omega_\rho \quad , \qquad \text{-----} \ (5.1.38)$$

$$U_i = b_{i\rho}(V_\rho + \epsilon_{\rho\alpha\beta}\omega_\alpha P_\beta) \quad . \qquad \text{-----} \ (5.1.39)$$

The dual momentum referred to the mass center is

$$\hat{H}_i = MV_i + \varepsilon H_i \quad , \qquad \text{-----} \ (5.1.40)$$

where the dual part, $H_i = I_i\omega_i$, is the angular momentum referred to the mass center. The dual vector equation of motion may be written as

$$\hat{G}_1 = \dot{\hat{H}}_1 + \epsilon_{1\alpha\beta}\,\omega_\alpha \hat{H}_\beta \quad , \qquad \text{-----} \ (5.1.41)$$

or

$$\hat{G}_1 = G_1 + \varepsilon N_1 \quad . \qquad \text{-----} \ (5.1.42)$$

The right hand side of Eq.(5.1.41) represents the rate of change of dual momentum with respect to the inertial space. To obtain the equations of motion for the offset unsymmetric gyroscope, we substitute Eqs.(5.1.35) and (5.1.36) into Eq.(5.1.42). Comparing the resultant equation with Eq.(5.1.42), we get

$$\hat{G}_1 = M\dot{V}_1 + \varepsilon(I_1\frac{d\omega_1}{dt} + \epsilon_{1\alpha\beta} I_\beta \omega_\alpha \omega_\beta) + M\epsilon_{1\alpha\beta}\,\omega_\alpha V_\beta \quad ,$$

$$= M(\dot{V}_1 + \epsilon_{1\alpha\beta}\,\omega_\alpha V_\beta) + \varepsilon(I_1\frac{d\omega_1}{dt} + \epsilon_{1\alpha\beta} I_\beta \omega_\alpha \omega_\beta \quad .$$

Therefore,

$$G_1 = M(\dot{V}_1 + \epsilon_{1\alpha\beta}\,\omega_\alpha V_\beta) \quad ,$$

$$N_1 = I_1\frac{d\omega_1}{dt} + \epsilon_{1\alpha\beta} I_\beta \omega_\alpha \omega_\beta \quad .$$

Expanding the above two equations, we then have the following six components :

$$G_1 = M[\ddot{z}S\theta S\psi + \ddot{x}C\psi + \ddot{\phi}(xC\theta S\psi + sS\theta C\psi) - s\ddot{\theta}S\psi + \dot{\phi}^2 (-xC\psi + sS\theta C\theta S\psi) + 2s\dot{\phi}\dot{\theta}C\theta C\psi - 2\dot{s}\dot{\theta}S\psi + 2\dot{x}\dot{\phi}C\theta S\psi + 2\dot{s}\dot{\phi}S\theta C\psi] \quad , \qquad \text{-----} \ (5.1.43)$$

$$G_2 = M[\ddot{z}S\theta C\psi - \ddot{x}S\psi + \ddot{\phi}(xC\theta C\psi - sS\theta S\psi) - s\ddot{\theta}C\psi + \dot{\phi}^2 (xS\psi + sS\theta C\theta C\psi) - 2s\dot{\phi}\dot{\theta}C\theta S\psi - 2\dot{s}\dot{\theta}C\psi + 2\dot{x}\dot{\phi}C\theta C\psi - 2\dot{s}\dot{\phi}S\theta S\psi] \quad , \qquad \text{-----} \ (5.1.44)$$

$$G_3 = M[\ddot{z}C\theta + \ddot{s} - x\ddot{\phi}S\theta - s\dot{\phi}^2S^2\theta - 2\dot{x}\dot{\phi}S\theta] \quad , \qquad \text{----- (5.1.45)}$$

$$N_1 = I_1\frac{d}{dt}(\dot{\phi}S\theta S\psi + \dot{\theta}C\psi) + (I_3 - I_2)[(\dot{\phi}C\theta + \dot{\psi})(\dot{\phi}S\theta C\psi - \dot{\theta}S\psi)] \quad , \qquad \text{----- (5.1.46)}$$

$$N_2 = I_2\frac{d}{dt}(\dot{\phi}S\theta C\psi + \dot{\theta}S\psi) + (I_1 - I_3)[(\dot{\phi}C\theta + \dot{\psi})(\dot{\phi}S\theta S\psi - \dot{\theta}C\psi)] \quad , \qquad \text{----- (5.1.47)}$$

$$N_3 = I_3\frac{d}{dt}(\dot{\phi}C\theta + \dot{\psi}) + (I_2 - I_1)[(\dot{\phi}S\theta S\psi + \dot{\theta}C\psi)(\dot{\phi}S\theta C\psi - \dot{\theta}S\psi)] \quad . \qquad \text{----- (5.1.48)}$$

The above two sets $\{G_1, G_2, G_3\}$ and $\{N_1, N_2, N_3\}$ are one set of forces and one set of moments exerted on the mass center C, respectively. The forces and moments exerted on the mass center are the resultant of the external dual force acting at various points of the system.

5.1.10 The equations of motion of the offset symmetric gyroscope

Most gyroscopic systems involve axial symmetric bodies. The offset symmetric gyroscope has a general representation as given in Eqs.(5.1.43) through (5.1.48). Let us take the offset symmetric gyroscopic system with c_3 as the axis of symmetry. The c_1 and c_2 may then be chosen arbitrarily in the plane passing through mass center C and normal to c_3. Thus, for convenience, we choose a frame $\{c\}_c$ attached to the inner gimbal such that c_1 is always parallel to b_1; then $\psi = 0$. Substituting $\psi = 0$ into Eqs.(5.1.31), (5.1.34) and (5.1.33), and using the properties

$$C\hat{\psi} = C\psi C\varepsilon s - S\psi S\varepsilon s = 1 ,$$
$$S\hat{\psi} = S\psi C\varepsilon s + S\varepsilon s C\psi = \varepsilon s ,$$

where $C\varepsilon s = 1$ and $S\varepsilon s = \varepsilon s$, we get the following transformation equations

$$a_1 = c_1 - \varepsilon s c_2 , \quad \text{-----} \ (5.1.49)$$

$$a_2 = c_1 \varepsilon s C\hat{\theta} + c_2 C\hat{\theta} - c_3 S\hat{\theta} , \quad \text{-----} \ (5.1.50)$$

$$a_3 = c_1 \varepsilon s S\hat{\theta} + c_2 S\hat{\theta} + c_3 C\hat{\theta} . \quad \text{-----} \ (5.1.51)$$

Substituting $I_2 = I_1$, $\psi = 0$ in Eqs.(5.1.43) through (5.1.48), we obtain the equations of motion for the offset symmetric gyroscope :

$$G_1 = M[\ddot{x} + s\ddot{\phi}S\theta - x\dot{\phi}^2 + 2s\dot{\phi}\dot{\theta}C\theta + 2\dot{s}\dot{\phi}S\theta] ,$$
$$\text{-----} \ (5.1.52)$$

$$G_2 = M[\ddot{z}S\theta + x\ddot{\phi}C\theta - s\ddot{\theta} + s\dot{\phi}^2 S\theta C\theta - 2\dot{s}\dot{\theta} + 2\dot{x}\dot{\phi}C\theta] ,$$
$$\text{-----} \ (5.1.53)$$

$$G_3 = M[\ddot{z}C\theta - x\ddot{\phi}S\theta + \ddot{s} - s\dot{\phi}^2 S^2\theta - 2\dot{x}\dot{\phi}S\theta - s\dot{\theta}^2 ,$$
$$\text{-----} \ (5.1.54)$$

$$N_1 = I_1(\ddot{\theta} - \dot{\phi}^2 S\theta C\theta) + I_3\dot{\phi}S\theta(\dot{\phi}C\theta + \dot{\psi}), \text{-----} \ (5.1.55)$$

$$N_2 = I_1(\ddot{\phi}S\theta + 2\dot{\phi}\dot{\theta}C\theta) - I_3\dot{\theta}(\dot{\phi}C\theta + \dot{\psi}), \text{-----} \ (5.1.56)$$

$$N_3 = I_3 \frac{d}{dt}(\dot{\phi}C\theta + \dot{\psi}) \quad , \qquad \text{-----} (5.1.57)$$

where $\dot{\psi}$ and $\dot{s}$ are, respectively, the angular and sliding velocities of the symmetric top about and along the spin axis c_3 attached to the moving inner gimbal. The above six equations represent a six degree-of-freedom system which is useful in the analysis of a variety of dynamical systems. Now, we will consider the following example which is the special case of the offset symmetric gyroscope.

Example Damped Spring-Mass System on a Periodically Moving Base

Fig.5.1J is a schematic diagram of damped spring-mass system mounted on a periodically moving base. The constraint conditions are $\theta = x = \dot{\phi} = \dot{\psi} = 0$ and $z = d\sin\omega t$; that is, A and B are coincident, and spring and dashpot are inserted between B and C. The reference frames are $\{a\}_A$ ($\phi = 0$, $z = d\sin\omega t$), $\{b\}_B$ ($\theta = 0$, $x = 0$) , $\{c\}_C$ ($\dot{\psi} = 0$, $s(t)$). We may write the force exerted on the mass center as

$$G_3 = -k(s-s_o) - Mg - c\dot{s} \quad , \qquad \text{-----} (5.1.58)$$

where k and s_o are, respectively, the modulus and the free length of the spring, and c is the damping coefficient. From Eq.(5.1.54) ,

$$G_3 = M(-d\omega^2\sin\omega t + \ddot{s}) \quad . \qquad \text{-----} (5.1.59)$$

From the Newton's second law, $\sum F = Ma$; and we can equate the right hand sides of Eqs.(5.1.58) and (5.1.59) to give

$$M\ddot{s} + ks + c\dot{s} = (ks_o - Mg) + Md\omega^2 \sin\omega t \quad \text{------ (5.1.60)}$$

The other five equations lead to $G_1 = G_2 = N_1 = N_2 = N_3 = 0$.

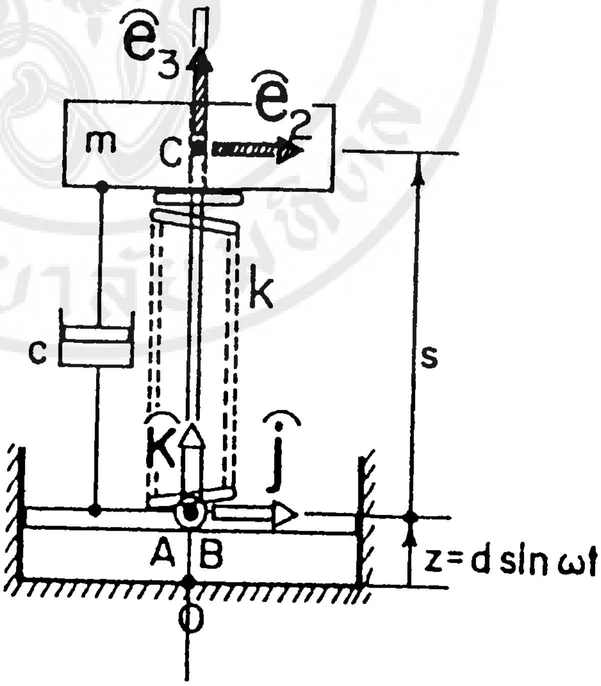


Fig.5.1.J A damped spring-mass system on a periodically moving base

Section 5.2 Application to Robotics : Robot Arm Kinematics

"A *robot* is a reprogrammable multi-functional manipulator designed to move materials, parts, tools, or specialized devices, through variable programmed motions for the performance of a variety of tasks." With this definition, as given by the Robot Institute of America, a robot must be controlled by computational programmes. Modern robotic systems consist of at least three major parts :

1. The manipulator, which is the mechanical moving structure,
2. The drives to actuate the joints of the manipulator,
3. The computer as a controller and storer of task programmes.

We will briefly survey the first part concerning the motion of a robot and the planning of manipulator trajectories. In general, the structure of a robot manipulator is composed of a main frame and a wrist with a tool at its end. The tool can be a welding head, a spray gun, a machining tool, or a gripper containing open-shut jaws, depending upon the specific application of the robot. The main frame is frequently referred to as the arm which consists of a sequence of mechanical links connected by joints. Now we will study the robot arm kinematics. ([2],[7],[22],[28],[32],[46],[68],[69],[73],[74])

5.2.1 Rotation matrices

Robot arm kinematics deals with the analytical study of the geometry of motion of a robot arm with respect to a fixed reference coordinate system as a function of time. The links of a robot arm may rotate and (or) translate with respect to a reference coordinate frame; the total spatial displacement of the end effector is due to the angular rotations and linear translations of the links. In 1955, Denavit and Hartenberg proposed a systematic and generalized approach of utilizing matrix algebra to describe and represent the spatial geometry of the links of a robot arm with respect to a fixed reference frame. We will now consider the rotational matrices, referring to Section 6 in Chapter II. The rotation matrix in terms of the Euler angles is

$$R(\phi n) = \exp(\tfrac{1}{2}\phi Z) , \qquad ----- (5.2.1)$$

where
$$Z = \begin{bmatrix} 0 & -n_z & n_y \\ n_z & 0 & -n_x \\ -n_y & n_x & 0 \end{bmatrix} ,$$

or
$$Z = n_x \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{bmatrix} + n_y \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ -1 & 0 & 0 \end{bmatrix} + n_z \begin{bmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} .$$

The above matrices can be expressed in terms of a matrix I, with components I_x, I_y, I_z, namely

$$I_x = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{bmatrix}, \quad I_y = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ -1 & 0 & 0 \end{bmatrix}, \quad I_z = \begin{bmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}.$$

Then Z is $\mathbf{n} \cdot I_\mu$, and Eq.(5.2.1) takes the form

$$R(\phi \mathbf{n}) = \exp(\tfrac{1}{2}\phi \mathbf{n} \cdot I_\mu) \text{ , where } \mu = x,y,z \qquad \text{,----- (5.2.2)}$$

or

$$R(\alpha x) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos(\frac{1}{2}\alpha) & -\sin(\frac{1}{2}\alpha) \\ 0 & \sin(\frac{1}{2}\alpha) & \cos(\frac{1}{2}\alpha) \end{bmatrix},$$

$$R(\beta y) = \begin{bmatrix} \cos(\frac{1}{2}\beta) & 0 & \sin(\frac{1}{2}\beta) \\ 0 & 1 & 0 \\ -\sin(\frac{1}{2}\beta) & 0 & \cos(\frac{1}{2}\beta) \end{bmatrix} \qquad \text{,----- (5.2.3)}$$

$$R(\gamma z) = \begin{bmatrix} \cos(\frac{1}{2}\gamma) & -\sin(\frac{1}{2}\gamma) & 0 \\ \sin(\frac{1}{2}\gamma) & \cos(\frac{1}{2}\gamma) & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

The depiction of the rotary joints of a robot in terms of the matrix representation helps ease the computation of the final resultant motion.

5.2.2 Direct kinematics

Kinematics is important in trajectory planning that is related to the velocities and accelerations of the hands and joints. In planning a movement of a manipulator, the Cartesian space of the manipulator is determined. In order to place the end of the manipulator at a given position in Cartesian space, we must solve for the corresponding joint angles, and thus obtain the final resultant motion. In the case of a two-link planar manipulator with rotary joints, both rotary joints (see Fig.5.2A) have the z axis as the axis of rotation, with θ_1 and θ_2 as the joint angles corresponding, respectively, to the first and the second joints. The lengths and masses for links 1 and 2 are l_1, m_1 and l_2, m_2, respectively.

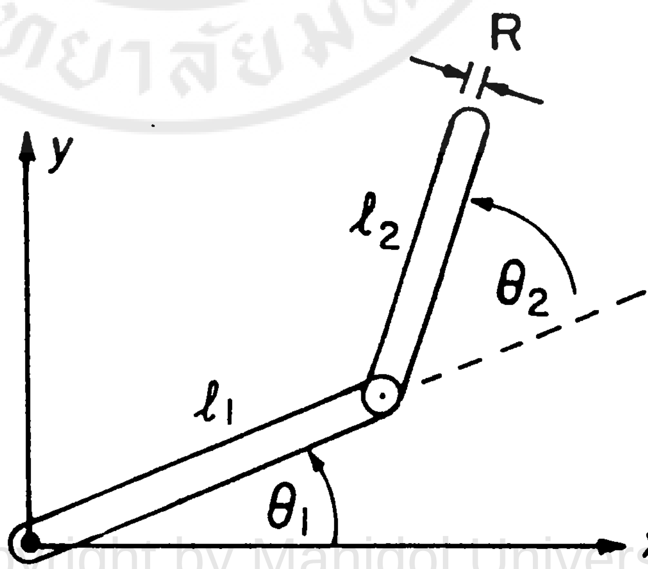


Fig. 5.2A A two-link planar manipulator with rotary joints

The joint positions are denoted by θ_1 and θ_2. The links have lengths l_1 and l_2, and both are assumed to be right circular cylinders of radius R.

Define P_i^* as the vector from the proximal to the distal joint of link i, where

$$P_1^* = l_1 \begin{bmatrix} \cos(\theta_1) \\ \sin(\theta_1) \end{bmatrix}, \qquad \text{----- (5.2.4)}$$

$$P_2^* = l_2 \begin{bmatrix} \cos(\theta_1 + \theta_2) \\ \sin(\theta_1 + \theta_2) \end{bmatrix}. \qquad \text{----- (5.2.5)}$$

Define P_i as the vector from the base to the distal joint of link i, which for the first link is $P_1 = P_1^*$ and for the second link is $P_2 = P_1^* + P_2^*$. The position of the end of the manipulator is given by $P_2=(x,y)$. Substituting Eq.(5.2.4) in Eq.(5.2.5), we get

$$\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} l_1\cos\theta_1 + l_2\cos(\theta_1 + \theta_2) \\ l_1\sin\theta_1 + l_2\sin(\theta_1 + \theta_2) \end{bmatrix}. \qquad \text{----- (5.2.6)}$$

Differentiating Eq.(5.2.4) to find relation between joint velocities and the Cartesian velocities of the manipulator tip, we have

$$\begin{bmatrix} \dot{x} \\ \dot{y} \end{bmatrix} = \begin{bmatrix} -l_1\sin\theta_1 - l_2\sin(\theta_1+\theta_2) & -l_2\sin(\theta_1+\theta_2) \\ l_1\cos\theta_1 + l_2\cos(\theta_1+\theta_2) & l_2\cos(\theta_1+\theta_2) \end{bmatrix} \begin{bmatrix} \dot{\theta}_1 \\ \dot{\theta}_2 \end{bmatrix},$$

----- (5.2.7)

where the 2×2 matrix is called the *Jacobian* J . By representing the vector of joint angles as $\theta = (\theta_1, \theta_2)$, this relation may be written as $P_2 = J\dot{\theta}$. The acceleration equations are most conveniently expressed in terms of the time derivatives of θ_1 and $\theta_1 + \theta_2$:

$$\begin{bmatrix} \ddot{x} \\ \ddot{y} \end{bmatrix} = \begin{bmatrix} -l_1\sin\theta_1 & -l_2\sin(\theta_1+\theta_2) \\ l_1\cos\theta_1 & l_2\cos(\theta_1+\theta_2) \end{bmatrix} \begin{bmatrix} \ddot{\theta}_1 \\ \ddot{\theta}_1+\ddot{\theta}_2 \end{bmatrix}$$

$$- \begin{bmatrix} l_1\cos\theta_1 & l_2\cos(\theta_1+\theta_2) \\ l_1\sin\theta_1 & l_2\sin(\theta_1+\theta_2) \end{bmatrix} \begin{bmatrix} \dot{\theta}_1^{\,2} \\ (\dot{\theta}_1+\dot{\theta}_2)^2 \end{bmatrix} . \text{-----} \quad (5.2.8)$$

5.2.3 Direct kinematics with rotation matrices

Regarding trajectory planning, the rotational matrices between adjacent coordinate systems in the joint space transformed from the Cartesian space. We thus would now like to derive the equations of motion for the two-link robot arm (see Fig. 5.2B) .

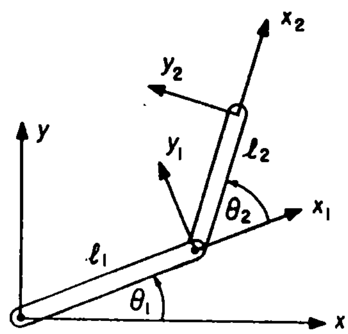


Fig. 5.2B The relation of points expressed in the link 2 coordinate system to the link 1 coordinate system

Projecting the axes x_1, y_1 onto the xoy coordinate systems, we have the following equations :

$$x = (x_1 + l_1)\cos\theta_1 - y_1\sin\theta_1 ,$$
$$y = (x_1 + l_1)\sin\theta_1 + y_1\cos\theta_1 ,$$

or, when written in matrix form,

$$\begin{bmatrix} x \\ y \end{bmatrix} = R(\theta_1)\begin{bmatrix} x_1 + l_1 \\ y_1 + 0 \end{bmatrix}$$

$$= R(\theta_1)\begin{bmatrix} x_1 \\ y_1 \end{bmatrix} + R(\theta_1)\begin{bmatrix} l_1 \\ 0 \end{bmatrix} , \qquad \text{----- (5.2.9)}$$

where

$$R(\theta_1) = \begin{bmatrix} \cos\theta_1 & -\sin\theta_1 \\ \sin\theta_1 & \cos\theta_1 \end{bmatrix} .$$

Projecting the axes x_2, y_2 onto the $x_1o_1y_1$ system in the same manner, we obtain

$$x_1 = (x_2 + l_2)\cos\theta_2 - y_2\sin\theta_2 ,$$
$$y_1 = (x_2 + l_2)\sin\theta_2 + y_2\cos\theta_2 ,$$

or $$\begin{bmatrix} x_1 \\ y_1 \end{bmatrix} = R(\theta_2)\begin{bmatrix} x_2 + l_2 \\ y_2 \end{bmatrix} . \qquad \text{----- (5.2.10)}$$

Substituting Eq.(5.2.10) into Eq.(5.2.9), we get

$$\begin{bmatrix} x \\ y \end{bmatrix} = R(\theta_1)R(\theta_2)\begin{bmatrix} x_2 + l_2 \\ y_2 \end{bmatrix} + R(\theta_1)\begin{bmatrix} l_1 \\ 0 \end{bmatrix} \qquad \text{----- (5.2.11)}$$

The end of the manipulator in link 2 coordinates (x_2, y_2) is (0,0). the position of the end in base coordinates, when substituted into Eq.(5.2.10), can be seen to yield Eq.(5.2.4) and Eq.(5.2.5), as is to be expected.

For more general manipulators, the link transformation matrices are more complex.

5.2.4 Planning straight-line trajectories using quaternions

It will be advantageous to transform the matrix representation of a rotation motion to the quaternion one. For a standard programme and operation of the robot manipulator, this will effectively reduce calculation time. Now, we will introduce the quaternions. A *quaternion* is a quadruple of ordered real numbers, s , a , b , c, associated, respectively, with four units : the real number +1 and three complex units , i , j , k, having cyclic permutations,

$$i^2 = j^2 = k^2 = -1 ,$$

$$i \times j = k , \quad j \times k = i , \quad k \times i = j ,$$

$$j \times i = -k , \quad k \times j = -i , \quad i \times k = -j .$$

A quaternion Q can be written as a four-component vector:

$$Q = s + ai + bj + ck = [s+V] = (s,a,b,c) = (s,V) .$$

The multiplication of two quaternions can be written as

$$Q_1 \times Q_2 = (s_1 + a_1 i + b_1 j + c_1 k) \times (s_2 + a_2 i + b_2 j + c_2 k)$$

$$= (s_1 s_2 - V_1 \cdot V_2 + s_2 V_1 + s_1 V_2 + V_1 \times V_2) .$$

In the case of vectors V_1, V_2 and scalar s_1, s_2, we can use the properties of quaternions

I Vector multiplication

$$V_1 \times V_2 = (0, V_1) \times (0, V_2) = (-V_1 \cdot V_2, V_1 \times V_2) \quad ,$$

II Scalar multiplication

$$s_1 \times s_2 = (s_1, 0) \times (s_2, 0) = (s_1 s_2, 0) \quad ,$$

III Vector-scalar multiplication

$$s \times V = (s, 0) \times (0, V) = (0, sV) \quad .$$

Rotation matrices are highly redundant, since they have nine entries, whereas an orthogonal set of three unit vectors can be completely specified by three numbers, an example of which is the three Euler angles that represent the rotation matrix. From Section 6, Chapter II,

$$R(\phi n) = \exp(\tfrac{1}{2} n \cdot I_\mu) \quad , \quad \text{where} \quad \mu = x, y, z.$$

The expression for the rotation, can be expanded accordingly in a power series. We then have

$$R(\phi n) = \cos(\tfrac{1}{2}\phi I) + \sin(\tfrac{1}{2}\phi I_\mu) \quad , \quad \text{where} \quad \mu = x, y, z.$$

From the above equation, we get the rotation in the quaternion form, that is,

$$\mathbf{R(\phi n)} = \mathbf{s} + \mathbf{V} \quad ,$$

where $s = \cos(\tfrac{1}{2}\phi)$ and $V = \sin(\tfrac{1}{2}\phi I_\mu)$, $\mu = x, y, z$ and

$$I_x = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{bmatrix}, \quad I_y = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ -1 & 0 & 0 \end{bmatrix}, \quad I_z = \begin{bmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}.$$

The rotation, written in the quaternion form, is denoted by the bracketed pair [s,V]. Similarly, we have $[\cos(\frac{1}{2}),\sin(\frac{1}{2}\phi I_{\mu})]$.

Example : A rotation of 90° about **k** followed by a rotation of 90° about **j** is represented by the quaternion product

$$(\cos45^\circ + \mathbf{j}\sin45^\circ)(\cos45^\circ + \mathbf{k}\sin45^\circ) = \frac{1}{2} + \frac{1}{2}\mathbf{i} + \frac{1}{2}\mathbf{j} + \frac{1}{2}\mathbf{k}$$

$$= \frac{1}{2} + \frac{(\mathbf{i}+\mathbf{j}+\mathbf{k})}{\sqrt{3}}(\frac{1}{2}\sqrt{3})$$

$$= \cos60^\circ + \sin60^\circ \frac{(\mathbf{i}+\mathbf{j}+\mathbf{k})}{\sqrt{3}},$$

$$\mathrm{Rot}(n,\theta) = \mathrm{Rot}\left[\frac{\mathbf{i}+\mathbf{j}+\mathbf{k}}{\sqrt{3}}, 120^\circ\right].$$

From the above, if rotations R_1 and R_2 are represented by quaternions Q_1 and Q_2, respectively, then the rotation $R_1 \times R_2$ will be represented by the quaternion $Q_1 \times Q_2$. It is clear, in this case, that the quaternion gives a much simpler representation. Thus one can change the representation for a rotation from the matrix form to the quaternion form, and vice versa. Quaternions offer another convenient representation for rotations and have been applied extensively to the analysis of kinematic linkages. For our purposes, the quaternion representation is more efficient than the matrix representation. Storage requirements are reduced, and calculations involving rotations can be done with fewer primitive operations (additions and multiplications) than are required if matrices are used (see Table 5.2C).

Table 5.2C Computational requirements of common rotational operations

Operation	Quaternion representation	Matrix representation
$R_1 \times R_2$	9 additions 16 multiplications	15 additions 24 multiplications
$R \cdot V$	12 additions 22 multiplications	6 additions 9 multiplications
$R \longrightarrow (n, \theta)$	4 multiplications 1 arctangent 1 square root	8 additions 10 multiplications 1 arctangent 2 square roots
$(n, \theta) \longrightarrow R$	4 multiplications 1 sine-cosine pair	10 additions 15 multiplications 1 sine-cosine pair
Convert to other representations	19 additions 9 multiplications	7 additions 5 multiplications 3 sine-cosine pairs 1 arctangent

Section 5.3 Application to Optics

In the case of *natural* or so-called *unpolarized* light, the instantaneous polarization fluctuates rapidly in a random manner. An ideal linear polarizing device, the behaviour of which can be conveniently approximated by a sheet of polaroid, allows only the component of the electric field in a given direction to be transmitted. In fact, we find that the transmitted light consists of constant polarization which fluctuates in a given direction. We now propose a mathematical description of the nonideal ("real") filter, as follows.

Consider the special case of a vector matrix

$$A = \begin{bmatrix} a & X \\ Y & b \end{bmatrix}$$

with $a,b \in \mathbb{R}$ and $X,Y \in \mathbb{V}$, where $\mathbb{V}$ is a three dimensional vector space. From the split octonions, the bases are

$$U_0 = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \quad U_0^* = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix},$$

$$U_1 = \begin{bmatrix} 0 & e_1 \\ 0 & 0 \end{bmatrix}, \quad U_1^* = \begin{bmatrix} 0 & 0 \\ -e_1 & 0 \end{bmatrix}.$$

If we let

$$W_0 = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \quad W_0^* = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix},$$

$$W_1 = \frac{1}{2}\begin{bmatrix} 1 & i \\ -i & 1 \end{bmatrix}, \qquad W_1^* = \frac{1}{2}\begin{bmatrix} 1 & -i \\ i & 1 \end{bmatrix},$$

where i is the imaginary unit, then $A = W_o + W_o^* + W_1 + W_1^*$. Thus the basis W a is linear combination of the vector matrices A.

Table 5.3A Multiplication table of W

	W_o	W_o^*	W_1	W_1^*
W_o	W_o	0	$\frac{1}{2}\begin{bmatrix} 1 & i \\ 0 & 0 \end{bmatrix}$	$\frac{1}{2}\begin{bmatrix} 1 & -i \\ 0 & 0 \end{bmatrix}$
W_o^*	0	W_o^*	$\frac{1}{2}\begin{bmatrix} 0 & 0 \\ -i & 1 \end{bmatrix}$	$\frac{1}{2}\begin{bmatrix} 0 & 0 \\ i & 1 \end{bmatrix}$
W_1	$\frac{1}{2}\begin{bmatrix} 1 & 0 \\ -i & 0 \end{bmatrix}$	$\frac{1}{2}\begin{bmatrix} 0 & i \\ 0 & 1 \end{bmatrix}$	W_1	0
W_1^*	$\frac{1}{2}\begin{bmatrix} 1 & 0 \\ i & 0 \end{bmatrix}$	$\frac{1}{2}\begin{bmatrix} 0 & -i \\ 0 & 1 \end{bmatrix}$	0	W_1^*

In Chapter II, Section 5, we know that

W_o is the transformation matrix of a linear polarizer with the transmission axis horizontal,

W_o^* is the transformation matrix of a linear polarizer polarizer with the transmission axis vertical,

W_1 is the transformation matrix of a right circular polarizer, and

W_1^* is the transformation matrix of a left circular polarizer.

If we substitute e_j (j = 1,2,3) for i so e_j is the basis of the three dimensional vector space, then the linear combination of the bases W becomes vector matrices. We can find the product of vector matrices by using Zorn's product, that is :

$$\begin{bmatrix} a & X \\ Y & b \end{bmatrix}\begin{bmatrix} c & U \\ V & d \end{bmatrix} = \begin{bmatrix} ac-(X\cdot V) & aU+dX+(Y\times V) \\ cY+bV+(X\times U) & bd-(Y\cdot U) \end{bmatrix} ,\text{-----} \quad (5.3.1)$$

where $\cdot$ and $\times$ are the usual dot and vector products of the three-dimensional vector space, respectively, and by using Myung's product, that is :

$$A * B = \frac{1}{2}[A,B] + \tau(A)B + \tau(B)A \quad ,$$

where $\tau(A) = 0$, $\tau(A) = ab$, and $\tau(A) = |X||Y|$. If we set the magnitude of X, Y, U, and V approaching 1, and set X_1, X_2, X_3 for having values just slightly different from one another, then the products of

$$Z_o = \begin{bmatrix} 1 & \underline{0} \\ \underline{0} & 0 \end{bmatrix} , \quad Z_o^* = \begin{bmatrix} 0 & \underline{0} \\ \underline{0} & 1 \end{bmatrix} ,$$

$$Z_1 = \frac{1}{2}\begin{bmatrix} 1 & X \\ Y & 1 \end{bmatrix} , \quad Z_1^* = \frac{1}{2}\begin{bmatrix} 1 & -X \\ Y & 1 \end{bmatrix} ,$$

where $\underline{0}$ is zero vector, have the same results as in table 5.2A, that is :

Table 5.3B Multiplication table of Z

	Z_o	Z_o^*	Z_1	Z_1^*
Z_o	Z_o	0	$\frac{1}{2}\begin{bmatrix} 1 & X \\ \underline{0} & 0 \end{bmatrix}$	$\frac{1}{2}\begin{bmatrix} 1 & -X \\ \underline{0} & 0 \end{bmatrix}$
Z_o^*	0	Z_o^*	$\frac{1}{2}\begin{bmatrix} 0 & \underline{0} \\ -Y & 1 \end{bmatrix}$	$\frac{1}{2}\begin{bmatrix} 0 & \underline{0} \\ Y & 1 \end{bmatrix}$
Z_1	$\frac{1}{2}\begin{bmatrix} 1 & \underline{0} \\ -Y & 0 \end{bmatrix}$	$\frac{1}{2}\begin{bmatrix} 0 & X \\ \underline{0} & 1 \end{bmatrix}$	Z_1	0
Z_1^*	$\frac{1}{2}\begin{bmatrix} 1 & \underline{0} \\ Y & 0 \end{bmatrix}$	$\frac{1}{2}\begin{bmatrix} 0 & -X \\ \underline{0} & 1 \end{bmatrix}$	0	Z_1^*

We now derive the product $Z_1 \cdot Z_1$:

$$\frac{1}{2}\begin{bmatrix} 1 & \mathbf{X} \\ -\mathbf{Y} & 1 \end{bmatrix} \cdot \frac{1}{2}\begin{bmatrix} 1 & \mathbf{X} \\ -\mathbf{Y} & 1 \end{bmatrix} = (1/4)\begin{bmatrix} 1+(\mathbf{X}\cdot\mathbf{Y}) & \mathbf{X}+\mathbf{X}+(\mathbf{Y}\times\mathbf{Y}) \\ -\mathbf{Y}-\mathbf{Y}+(\mathbf{X}\times\mathbf{X}) & 1+(\mathbf{Y}\cdot\mathbf{X}) \end{bmatrix}$$

$$= (1/4)\begin{bmatrix} 1+(X_1Y_1+X_2Y_2+X_3Y_3) & 2\mathbf{X} \\ -2\mathbf{Y} & 1+(X_1Y_1+X_2Y_2+X_3Y_3 \end{bmatrix} . \text{------ } (5.3.2)$$

Let $|\mathbf{X}| \approx |\mathbf{Y}| \approx 1$ and $X_1 \approx X_2 \approx X_3 \approx Y_1 \approx Y_2 \approx Y_3$, then

$$\sqrt{X_1^2 + X_2^2 + X_3^2} = 1 ,$$
$$3X_1^2 \approx 1 . \quad \text{----- } (5.3.3)$$

From Eq.(5.3.1) , we get

$$Z_1 \cdot Z_1 \approx \frac{1}{2}\begin{bmatrix} \frac{1}{2}(1+3X_1^2) & \mathbf{X} \\ -\mathbf{Y} & \frac{1}{2}(1+3X_1^2) \end{bmatrix}$$

$$\approx \frac{1}{2}\begin{bmatrix} 1 & \mathbf{X} \\ -\mathbf{Y} & 1 \end{bmatrix} , \text{ from Eq.(5.3.3) } .$$

We can thus find

$$Z_1 \cdot Z_1^* = \frac{1}{2}\begin{bmatrix} 1 & \mathbf{X} \\ -\mathbf{Y} & 1 \end{bmatrix} \cdot \frac{1}{2}\begin{bmatrix} 1 & -\mathbf{X} \\ \mathbf{Y} & 1 \end{bmatrix}$$

$$= (1/4)\begin{bmatrix} 1-(\mathbf{X}\cdot\mathbf{Y}) & -\mathbf{X}+\mathbf{X}-(\mathbf{Y}\times\mathbf{Y}) \\ -\mathbf{Y}+\mathbf{Y}-(\mathbf{X}\times\mathbf{X}) & 1-(\mathbf{Y}\cdot\mathbf{X}) \end{bmatrix}$$

$$\approx (1/4)\begin{bmatrix} 1-3X_1^2 & 0 \\ 0 & 1-3X_1^2 \end{bmatrix} \approx 0 ,$$

Similarly,

$$Z_1^{*} \cdot Z_1 \approx 0 \quad ,$$

$$Z_1^{*} \cdot Z_1^{*} \approx \frac{1}{2}\begin{bmatrix} 1 & -X \\ Y & 1 \end{bmatrix} \quad .$$

In substituting X for i, which has magnitude approaching 1 , we suggest that the polarization of the plane harmonic electromagnetic wave has small random fluctuations. The orientation of the electric field is not constant, but slightly varying. The detailed results are shown in Appendix B.

CHAPTER VI

DISCUSSION AND CONCLUSION

In the preceding chapters, we have examined some of the concepts and applications of vector matrices, hypercomplex numbers, and dual numbers. At this stage, a critical assessment of these, and a few of the ways in which the subject is likely to develop in the near future, will be considered. The general usefulness and effectiveness of vector matrices, hypercomplex numbers, dual numbers, and other hypernumbers, in solving mathematical problems in physics and engineering, will also be further discussed.

Section 6.1 Discussion

The first chapter of this thesis introduced general knowledge concerning the concept of *number*, from natural numbers to hypernumbers. In addition, the p-adic, nonstandard, and surreal (Conway) numbers were presented. All of this is certainly a significant and well-established part of pure mathematics; but much of it is still beyond the range of knowledge and interest of most applied mathematicians, physical scientists, and engineers. The great extension of our ideas of *generalized* numbers, with their algebraic, topological, geometric and analytic structures, awaits future applications.

Chapter II emphasized the algebraic structures of physics. The great importance and the wide-ranging roles of the matrix groups, such as GL(n,K), SL(n,K), and U(n), in various branches of physics, have already been well recognized. The connection of the hypercomplex numbers with the matrix groups, however, is not yet so familiar to most physicists. Lie algebras and Lie groups were first described in Section 2.2 . Section 2.4 considered some of their properties in detail, focussing on the roles of Poisson algebras and Heisenberg groups in classical and quantum mechanics. The wider use of hypercomplex numbers, and other generalized and extended number systems, in physics, has required some specialized knowledge of nonassociative algebraic structures that include Jordan, alternative, and nilpotent algebras, in addition to that of the Lie algebras, with which physicists have now become quite familiar. These were considered in Section 2.3 .

In the case of the calculus of polarization, the Stokes parameters clearly provide a simple and elegant formulation of the algebraic and analytic description of polarized electromagnetic radiation in classical electrodynamics. Moreover, the Stokes parameters, and the related Jones and Mueller matrices, are known to be directly related to the elements of the density matrix in quantum mechanics. The algebraic structures of the calculus polarization, and of the Jones and Mueller calculus were discussed in Section 2.5 . Also in Chapter II, the principles and the procedures of transforming spinors and twistors were presented by using stereographic projection and the Riemann sphere. Both the algebraic and the geometrical aspects of the holomorphic transformations of spinors and twistors were emphasized. The present use of these concepts and results in quantum and relativistic physics was briefly indicated in Section 2.6 . In this connection, the great importance of the special unitary groups can be easily recognized.

The core of Chapter III was concerned with the generalized vector matrices, bimatrices, and octonions. Some general features of the octonions and Zorn's vector matrices were first described. The study was then focussed on the power-associative products on the ordinary and the split octonions that can give rise to the properties of flexibility, antiflexibility, third and fourth power-associativity. Further considerations of the vector matrices led naturally to the concepts of graded Lie-admissibility and the bimatrices that are the combined parts of hypernumbers. The recent successful introduction of graded Lie algebras, and of Lie superalgebras and supergroups into particle and nuclear physics is a clear indication of the need to widen the basis of algebraic and geometric thinking in theoretical and mathematical physics so as to include generalized vector matrices, bimatrices, octonions, etc., within its domain.

In Chapter IV, the representation of generalized vector matrices was illustrated in an "8-dimensional" system, by showing the combination of two ordinary three-dimensional vectors and two scalars. It is expected that this simplified representation should lead to a readier understanding and to the development of some intuition for the 8-dimensional problems of interest in mathematical and theoretical physics.

Chapter V considered three applications of generalized vector matrices, hypernumbers, and dual numbers in physical and engineering problems. The dual numbers have been used in the theory of the offset unsymmetrical gyroscope in sorting out the general equations of motion. In particular, the roles of the dual-transformation matrix and of the dual-Euler angles and equations should be noted. The analysis of an offset unsymmetric gyroscope with oblique rotor has demonstrated the useful application of 3x3 matrices with dual-number elements for the description of the kinematic relations within a system involving a number of reference frames with no common origin. Results obtained thus far have led us to believe that the 3x3 dual-transformation matrices, in view of their orthogonality properties and ease of adaptation to tensor notation, will offer a meaningful alternative in the analytical treatment of the mechanics of a system of rigid bodies in spatial motion. It should be noted that the dual-vector calculus applied here for the acceleration analysis of the RCCC mechanism, a spatial four-link mechanism with one revolute pair and three cylindrical pairs, may be applied to similar analysis of other types of spatial mechanisms. It is thus hoped that dual-vector calculus may soon find wider applications in the kinematic analysis of mechanisms.

In robotics, the development of the trajectory planning is very useful in increasing the efficiency of the manipulator path control algorithms. The method of quaternions, which offers an alternative representation of rotations, has led to simple formulae for the rotations of vectors, and to the composition of rotations. It is well recognized that the method of rotation matrices, which is at present widely used in robotics, is rather expensive computationally, and is vulnerable to unexpected difficulties where degeneracies or joint limits are encountered. In many situations, however, the number of the arithmetical operations required for the kinematic solution of a robot manipulator can be significantly decreased by using the quaternion method. Certainly, the use of quaternions, dual numbers, and other hypercomplex numbers, in robotics, deserves further attention from applied mathematicians and engineers.

As regards some mathematical problems in optics considered in Chapter V, the assumption that the Jones and the Mueller matrices may conveniently be replaced by generalized vector matrices in the description of polarization of electromagnetic waves, should find wider applications. Jones vectors are added vectorially in calculating the result of two or more waves of given polarizations. But the Jones matrix is the transformation matrix of the linear optical element. If, in allowing for some possible nonlinearity, we change the off-diagonal elements of the generalized vector matrices to unit three-dimensional vectors with arbitrary orientations, we still find that the Zorn product of two vector matrices gives quite a satisfactory description of the physical situations according to our assumption. Also, it should be noted that the original Jones calculus is of use only for computing results with light that is initially polarized in some way, but the vector matrices can describe the mixture of polarized light and a little of unpolarized light present as "fluctuations". It is thus quite likely that generalized vector matrices may soon find further applications in optics.

Section 6.2 Conclusion

"Number is the bond of the eternal continuance of things." Thus was the view of the Pythagorean, Philolaus (flourished *circa* 475 B.C.) . For millennia, *numbers* have played important roles in human societies. They are indeed an indispensable tool of civilization. A primitive society needs at least some notion of *natural* numbers and simple arithmetical operations for reckoning and for the calendar. It is an accepted fact that our modern technological and informational society needs *real* and *complex* numbers, and calculus in terms of them. But the concept of number in contemporary mathematics has gone much beyond this. What should then be the relevance of the "higher" kinds of number to science and technology ?

In this thesis, we have attempted to show how some of the generalized ideas of number should have important and interesting applications in physical science and in engineering. *Hypercomplex* numbers, and some other types of *hypernumbers*, have already been well accepted in mathematical and theoretical physics. But, up till now, a very limited number of applied scientists and engineers know anything about them. We have illustrated how hypercomplex numbers, dual numbers, and some generalized vector matrices that are closely related to them, can be applied to the solution of a few interesting mathematical problems in the dynamics of gyroscopes, in the kinematics of robot-arm manipulators, and in the theoretical treatment of polarization in optics. However, we have not considered the application of many other important types of number. Yet it should be noted that p-adic numbers and p-adic analysis are now rapidly entering mathematical physics, and have already provided many interesting alternative "models", specifically for the situations where the *Archimedean order* of the physical world is in doubt. Nonstandard analysis based on *hyperreal* numbers, and the use of *surreal* (Conway) numbers, for example, in the theory of physical measurements, are also leading us towards a new conception of "continuity" in physics.

A proper understanding of the *hypernumbers* beyond the *hypercomplex* quaternions needs a considerable knowledge of algebra. In this thesis, however, we have presented only the rudiments and the essentials. Also, it must be realized that *algebraic conceptualization* is only one of the principal modes of mathematical thinking. We have not adequately dealt with the *geometrical*, *topological* and *axiomatic* aspects of hypernumbers. Nevertheless, a simple pictographical computer method has been introduced to aid the "visualization" of the generalized vector matrices.

In the case of the *complex* numbers, it was mainly due to their *geometrical* interpretation by Carl Friedrich Gauss (1777-1855) that the aura of mysticism was much removed from the "imaginary" numbers. He expressed the attitude of his contemporaries to these numbers as follows : "... but these imaginary numbers, as opposed to real quantities -- formerly, and even now occasionally, though improperly called *impossible* — have been merely tolerated rather than given full citizenship and appear therefore more like a game played with symbols devoid of content in itself, to which one refrains absolutely from ascribing any visualizable substratum. In saying this one has no wish to belittle the rich tribute which this play with symbols has contributed to the treasury of relations between real numbers." (quoted by R. Remmert in [22], p.61) Gauss emphasized the important role of a suitable *geometrical representation* in making the abstract conception of number more familiar to us. Reminising, after 1831, he said : "...so long as imaginary quantities were still based on a fiction, they were not, so to say, fully accepted in mathematics but were regarded rather as something to be tolerated; they remained far from being given the same status as real quantities. There is no longer any justification for such discrimination now that the metaphysics of imaginary numbers has been put in a true light and that it has been shown that they have just as good a real objective meaning as the negative numbers." (quoted by R. Remmert in [22], p.62)

Once the "imaginary" and *complex* numbers had been formally accepted in mathematics, it took a few centuries to find such a geometrical interpretation that would make them suitable for wide applications in physics. It took nearly a century longer before they entered electrical engineering, the technological consequence of which has since made many important contributions to our present civilization. But, in mathematics, once the mystery had been largely removed from the "imaginary", the way was open for Hamilton, Graves, Cayley, Clifford, and others, to *discover*, or, rather, to "invent" the *hypercomplex* numbers, and to even go beyond them to the more general *hypernumbers*. Much is still to be done to properly "geometrize" them, and to make them more familiar to applied scientists and engineers. Perhaps, like vectors, tensors, matrices, and groups, it will then be not too long before these hypernumbers will permeate science and engineering. It is rather appropriate to end this thesis with the dictum of Richard Dedekind (1831-1916), who contributed much to our present conception of numbers, viz.,

> " Numbers are free creations of the human intellect, they serve as a means of grasping more easily and more sharply the diversity of things. "

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APPENDIX A

COMPUTER PROGRAMME

Turbo Pascal Language (Version 5.0)

```
Program MyungAndZornGraphic;
Uses    Crt,Graph;
Type    VecMat    = Array [1..20,0..2,1..3] of Real;
        Mat       = Array [0..2,1..3] of Real;
        Vec       = Array [1..3] of Real;
Var     dataX,dataY,dataZ : VecMat;
        X,Y,Z             : Mat;
        i,n,j,k,l         : Integer;
        ans               : Char;
        ref               : Boolean;

Procedure DotVector(Var X,Y       : Mat;
                    Var sumx,sumy : Real);

(* ************************************************* *)
(*  This procedure multiplies two vectors by using   *)
(*  scalar product; the result is a scalar.          *)
(* ************************************************* *)

Var    j           : Integer;
       spq,sxv,syu : Real;

Begin   { Procedure  DotVector }
    sumx := 0;
    sumy := 0;
    For j := 1 to 3 Do
       Begin
          sxv  := X[1,j] * Y[2,j];
          sumx := sumx + sxv;
          syu  := X[2,j] * Y[1,j];
          sumy := sumy + syu;
       End;
End;    { Procedure  DotVector }


Procedure CrossVector(Var X,Y : Mat;
                      Var vpq : Vec;
                            j : Integer);

(* ************************************************* *)
(*  This procedure multiplies two vectors by using   *)
(*  vector product; the result is a vector.          *)
(* ************************************************* *)

Var    pq1p,pq1n,pq2p,pq2n,pq3p,pq3n : Real;

Begin   { Procedure  CrossVector }
    pq1p   := X[j,2] * Y[j,3];
    pq1n   := X[j,3] * Y[j,2];
    vpq[1] := pq1p - pq1n;
```

```
      pq2p   := X[j,3] * Y[j,1];
      pq2n   := X[j,1] * Y[j,3];
      vpq[2] := pq2p - pq2n;
      pq3p   := X[j,1] * Y[j,2];
      pq3n   := X[j,2] * Y[j,1];
      vpq[3] := pq3p - pq3n;
End;    { Procedure  CrossVector }


Procedure Magnitude(Var dataZ : VecMat;
                    Var mW,mZ : Real;
                            i : Integer);

(* ************************************************ *)
(*  This procedure finds the magnitude of a vector.  *)
(* ************************************************ *)

Var     j                      : Integer;
        SqVecW,SqVecZ,sumW,sumZ : Real;

Begin   { Procedure  Magnitude }
     sumW := 0;
     sumZ := 0;
     For j := 1 to 3 Do
        Begin
          SqVecW := Sqr(dataZ[i,1,j]);
          sumW := sumW + SqVecW;
          SqVecZ := Sqr(dataZ[i,2,j]);
          sumZ := sumZ + SqVecZ;
        End;
     mW := Sqrt(sumW);
     mZ := Sqrt(sumZ);
End;    { Procedure  Magnitude }


Procedure GetData (Var dataX,dataY : VecMat;
                   Var           n : Integer);

(* ************************************************ *)
(*   This procedure reads data from file.           *)
(* ************************************************ *)

Var     Fvar    :  Text;
        i,j,k   :  Integer;

Begin   { Procedure  GetData }
     Assign(Fvar,'B:NewData.dat');
     Reset(Fvar);
     While not EOF(Fvar) Do
        Begin
```

```
            While not EOLn(Fvar) Do
              Begin
                Read(Fvar,i,dataX[i,0,1],dataX[i,0,2]);
                For j := 1 to 3 Do  Read (Fvar,dataX[i,1,j]);
                For j := 1 to 3 Do  Read (Fvar,dataX[i,2,j]);
                Read(Fvar,dataY[i,0,1],dataY[i,0,2]);
                For j := 1 to 3 Do  Read (Fvar,dataY[i,1,j]);
                For j := 1 to 3 Do  Read (Fvar,dataY[i,2,j]);
              End;
            Readln(Fvar);
          End;
        n := i;
        Close(Fvar);
 End;    { Procedure  GetData }


 Procedure KeepData  (Var dataX,dataY,dataZ : VecMat;
                       Var              n : Integer);

 (* ****************************************** *)
 (*  This procedure puts the results into the file.  *)
 (* ****************************************** *)

 Var     Fname    :  Text;
         i,j,k    :  Integer;
         mw,mz    :  Real;

 Begin   { Procedure  KeepData }
     Assign(Fname,'B:OutZorn.dat');
     Rewrite(Fname);
     Writeln(Fname,'This data is Zorn's vector matrices.');
     Writeln(Fname);
     Writeln(Fname,'A realization of the split octonion algebra');
     Writeln(Fname,'is via the Zorn's vector matrices');
     Writeln(Fname);
     Writeln(Fname,'    ¦ a  X^ ¦ , ');
     Writeln(Fname,'    ¦ Y^ b  ¦   ');
     Writeln(Fname);
     Writeln(Fname,'where a and b are scalars and X^ and Y^ are ');
     Writeln(Fname,'3-dimensional vectors, with the product defined as');
     Writeln(Fname);
     Writeln(Fname,'¦ a  X^ ¦ ¦ c  U^ ¦  =  ¦ ac + X^V^  aU^+ dX^- [Y^x V^] ¦');
     Writeln(Fname,'¦ Y^ b  ¦ ¦ V^ d  ¦     ¦ cY^+ bV^+ [X^x U^]  Y^U^+ bd  ¦');
     Writeln(Fname);
     Writeln(Fname,'x denotes the usual vector product.');
     Writeln(Fname);
     Writeln(Fname,  'and let');
     Writeln(Fname,'   ¦ e  W^ ¦  =  ¦ ac + X^V^   aU^+ dX^- [Y^x V^] ¦');
     Writeln(Fname,'   ¦ Z^ f  ¦     ¦ cY^+ bV^+ [X^x U^]  Y^U^+ bd  ¦');
     Writeln(Fname);
```

```
    Writeln(Fname);
    Writeln(Fname);
    Writeln(Fname,'The following table is values of scalars ');
    Writeln(Fname,'[a,b,c,d,e,f]   | vectors [X^,Y^,U^,V^,W^,Z^]');
    Writeln(Fname);
    Writeln(Fname,'  N     a      b              X^            Y^');
    Writeln(Fname,'        c      d              U^            V^');
    Writeln(Fname,'        e      f              W^            Z^');
    Writeln(Fname);
    Writeln(Fname);
    For i := 1 to n Do
      Begin
        Magnitude(dataZ,mw,mz,i);
        Write(Fname,i:3,dataX[i,0,1]:8:3,dataX[i,0,2]:8:3,'   [');
        For j := 1 to 3 Do  Write(Fname,dataX[i,1,j]:8:3);
                            Write(Fname,'  ]   [');
        For j := 1 to 3 Do  Write(Fname,dataX[i,2,j]:8:3);
                            Writeln(Fname,' ]');
                            Writeln(Fname);
        Write(Fname,'   ',dataY[i,0,1]:8:3,dataY[i,0,2]:8:3,'   [');
        For j := 1 to 3 Do  Write(Fname,dataY[i,1,j]:8:3);
                            Write(Fname,'  ]   [');
        For j := 1 to 3 Do  Write(Fname,dataY[i,2,j]:8:3);
                            Writeln(Fname,' ]');
                            Writeln(Fname);
        Write(Fname,'   ',dataZ[i,0,1]:8:3,dataZ[i,0,2]:8:3,'   [');
        For j := 1 to 3 Do  Write(Fname,dataZ[i,1,j]:8:3);
                            Write(Fname,'  ]   [');
        For j := 1 to 3 Do  Write(Fname,dataZ[i,2,j]:8:3);
                            Writeln(Fname,' ]');
                            Writeln(Fname);
        Writeln(Fname,'    | W^| = ',mw:8:3,'      | Z^| = ',mz:8:3);
        Writeln(Fname);
        Writeln(Fname);
    End;
    Close(Fname);
End;    { Procedure  KeepData }


Procedure ChangeVecMatToMat(Var dataX : VecMat;
                            Var     X : Mat;
                                    i : Integer);

(* ************************************************* *)
(*   This procedure changes a vector matrix to a     *)
(*   set of vectors.                                 *)
(* ************************************************* *)

Var      j,k : Integer;
```

```
Begin   { Procedure  ChangeVecMatToMat }
    X[0,1] := dataX[i,0,1];
    X[0,2] := dataX[i,0,2];
    For  j := 1 to 2 Do
       Begin
          For k := 1 to 3 Do
          X[j,k] := dataX[i,j,k];
       End;
End;    { Procedure ChangeVecMatToMat }


Procedure ChangeMatToVecMat(Var Z     : Mat;
                            Var dataZ : VecMat;
                                    i : Integer);

(* ************************************************* *)
(*   This procedure changes the set of vectors to    *)
(*   a vector matrix.                                *)
(* ************************************************* *)

Var     j,k : Integer;

Begin   { Procedure ChangeMatToVecMat }
    dataZ[i,0,1] := Z[0,1];
    dataZ[i,0,2] := Z[0,2];
    For j := 1 to 2 Do
       Begin
          For k := 1 to 3 Do
          dataZ[i,j,k] := Z[j,k];
       End;
End;    { Procedure  ChangeMatToVecMat }


Procedure Zorn(Var X,Y,W : Mat);

(* ************************************************* *)
(*   This procedure multiplies two vector            *)
(*   matrices by using Zorn product.                 *)
(* ************************************************* *)

Var     vau,vdx,vyv,vcy,vbv,vxu        : Vec;
        sac,sbd,sxv,syu,sumxv,sumyu    : Real;
        j,k                            : Integer;

Begin   { Procedure  Zorn }
    sac := X[0,1] * Y[0,1];
    sbd := X[0,2] * Y[0,2];
    DotVector(X,Y,sumxv,sumyu);
    CrossVector(X,Y,vxu,1);
    CrossVector(X,Y,vyv,2);
```

```
    For j := 1 to 3 Do
       Begin
          vau[j] := X[0,1] * Y[1,j];
          vdx[j] := Y[0,2] * X[1,j];
          vcy[j] := Y[0,1] * X[2,j];
          vbv[j] := X[0,2] * Y[2,j];
          W[1,j] := vau[j] + vdx[j] + vyv[j];
          W[2,j] := vcy[j] + vbv[j] + vxu[j];
       End;
    W[0,1] := sac + sumxv;
    W[0,2] := sbd + sumyu;
End;    { Procedure  Zorn }


Procedure Mag(Var X     : Mat;
              Var mW,mZ : Real);

(* ********************************************** *)
(*  This procedure finds the magnitude of a vector.  *)
(* ********************************************** *)

Var     j                        : Integer;
        SqVecW,SqVecZ,sumW,sumZ : Real;

Begin   { Procedure  Mag }
    sumW := 0;
    sumZ := 0;
    For j := 1 to 3 Do
       Begin
          SqVecW := Sqr(X[1,j]);
          sumW := sumW + SqVecW;
          SqVecZ := Sqr(X[2,j]);
          sumZ := sumZ + SqVecZ;
       End;
    mW := Sqrt(sumW);
    mZ := Sqrt(sumZ);
End;    { Procedure  Mag }


Function Tau(check : Integer;X : Mat) : Real;

(* ********************************************** *)
(*   This is the linear functional TAU.           *)
(* ********************************************** *)

Var     j          : Integer;
        mX,mY,sum : Real;

Begin   { Function  Tau }
    Mag(X,mX,mY);
```

```
    Case check of
        1 : Tau := 0;
        2 : Tau := X[0,1] * X[0,2];
        3 : Tau := mX * mY;
        4 : Tau := Sqrt(Sqr(X[0,1]) + Sqr(X[0,2])
                   + Sqr(mX) + Sqr(mY));
    End;
End;    { Function Tau }


Procedure VecMulScalar(Var TauX    : Real;
                       Var Y,TauXY : Mat);

(* ********************************************** *)
(*   Vector multiplied by scalar.                 *)
(* ********************************************** *)

Var     j,k  : Integer;

Begin   { Procedure  VecMulScalar }
    TauXY[0,1] := TauX * Y[0,1];
    TauXY[0,2] := TauX * Y[0,2];
    For j := 1 to 2 Do
       Begin
         For k := 1 to 3 Do
           TauXY[j,k] := TauX * Y[j,k];
       End;
End;    { Procedure  VecMulScalar }


Procedure Calculation(Var X,Y,XstarY : Mat);

(* ********************************************** *)
(*   This procedure multiplies two vector matrices *)
(*   by using Myung product.                       *)
(* ********************************************** *)

Var      XY,YX ,XcommuteY,TauXY,TauYX : Mat;
         TauX,TauY                    : Real;
         j,k                          : Integer;

Begin   { Procedure  Calculation }
    Zorn(X,Y,XY);
    Zorn(Y,X,YX);
    TauX := Tau(1,X);
    TauY := Tau(1,Y);
    VecMulScalar(TauX,Y,TauXY);
    VecMulScalar(TauY,X,TauYX);
    For k := 1 to 2 Do
       Begin
```

```
            XcommuteY[0,k] := 0.5 * (XY[0,k] - YX[0,k]);
            XstarY[0,k] := XcommuteY[0,k] + TauXY[0,k] + TauYX[0,k];
          End;
      For j := 1 to 2 Do
        Begin
          For k := 1 to 3 Do
            Begin
              XcommuteY[j,k] := 0.5 * (XY[j,k] - YX[j,k]);
              XstarY[j,k] := XcommuteY[j,k] + TauXY[j,k] + TauYX[j,k];
            End;
        End;
End;    { Procedure  Calculation }


Procedure CheckEqual(Var P,Q     : Mat;
                     Var compare : Boolean);

(* ********************************************** *)
(*   This procedure checks the equality of two     *)
(*   vector matrices.                              *)
(* ********************************************** *)

Var     j,k   : Integer;
        check : Boolean;

Begin   { Procedure  CheckEqual }
    compare := True;
    check := True;
    If (P[0,1] - Q[0,1] < 0.00001) and (P[0,2] - Q[0,2] < 0.00001)
       Then check := True
    Else check := False;
    compare := compare and check;
    For j := 1 to 2 Do
      Begin
        For k := 1 to 3 Do
          Begin
            If P[j,k] - Q[j,k] < 0.00001 Then check := True
            Else check := False;
            compare := compare And check;
          End;
      End;
End;    { Procedure  CheckEqual }


Procedure CheckFlexible(var X,Y : Mat;
                               i : Integer);

(* ********************************************** *)
(*   This procedure checks the flexibility of two  *)
(*   vector matrices.                              *)
```

```
(* ************************************************** *)

Var     YX,XmulYX,XY,XYmulX : Mat;
        j,k                 : Integer;
Begin   { Procedure  CheckFlexible }
    Calculation(Y,X,YX);
    Calculation(X,YX,XmulYX);
    Calculation(X,Y,XY);
    Calculation(XY,X,XYmulX);
    CheckEqual(XmulYX,XYmulX,ref);
    If ref = True Then Writeln('Data set ',i,' is Flexible')
    Else Writeln('Data set ',i,' is not Flexible');
    Write(XmulYX[0,1]:8:3,XmulYX[0,2]:8:3,'   [');
    For j := 1 to 3 Do  Write(XmulYX[1,j]:8:3);
                        Write('  ]   [');
    For j := 1 to 3 Do  Write(XmulYX[2,j]:8:3);
                        Writeln('  ]');
    Write(XYmulX[0,1]:8:3,XYmulX[0,2]:8:3,'   [');
    For j := 1 to 3 Do  Write(XYmulX[1,j]:8:3);
                        Write('  ]   [');
    For j := 1 to 3 Do  Write(XYmulX[2,j]:8:3);
                        Writeln('  ]');
    Writeln;
    Writeln;
    Readln;
End;    { Procedure  CheckFlexible }


Procedure Associative(Var X      : Mat;
                          select : Integer);

(* ************************************************** *)
(*  This procedure checks the third and fourth        *)
(*  power associative products on the vector          *)
(*  matrices.                                         *)
(* ************************************************** *)

Var     XX,XmXX,XXmX,XXmXX,XpXmXX : Mat;
        XXmXpX,XpXXmX,XmXXpX      : Mat;
        j,k              : Integer;
        ref1,ref2,ref3   : Boolean;

Begin   { Procedure  Associative }
    Calculation(X,X,XX);
    Calculation(XX,X,XXmX);
    Calculation(X,XX,XmXX);
    If select = 3  Then
      Begin
        CheckEqual(XXmX,XmXX,ref);
        If ref = True Then
```

```
                Writeln('Data set ',i,' has 3rd power associative product')
            Else Writeln('Data set ',i,' has not 3rd power associative product');
            Write(XmXX[0,1]:8:3,XmXX[0,2]:8:3,'   [');
            For j := 1 to 3 Do  Write(XmXX[1,j]:8:3);
                                Write('  ]    [');
            For j := 1 to 3 Do  Write(XmXX[2,j]:8:3);
                                Writeln('  ]');
            Write(XXmX[0,1]:8:3,XXmX[0,2]:8:3,'   [');
            For j := 1 to 3 Do  Write(XXmX[1,j]:8:3);
                                Write('  ]    [');
            For j := 1 to 3 Do  Write(XXmX[2,j]:8:3);
                                Writeln('  ]');
            Writeln;
            Writeln;
            Readln;
         End
      Else
         Begin
            Calculation(XX,XX,XXmXX);
            Calculation(X,XmXX,XpXmXX);
            Calculation(XXmX,X,XXmXpX);
            Calculation(X,XXmX,XpXXmX);
            Calculation(XmXX,X,XmXXpX);
            CheckEqual(XXmXX,XpXmXX,ref1);
            CheckEqual(XXmXpX,XpXXmX,ref2);
            CheckEqual(XmXXpX,XpXXmX,ref3);
            If (ref1 And ref2 And ref3) = True
              Then  Writeln('Data set ',i,' has 4th power associative product')
            Else Writeln('Data set ',i,' has not 4th power associative product');
            Write(XXmXX[0,1]:8:3,XXmXX[0,2]:8:3,'   [');
            For j := 1 to 3 Do  Write(XXmXX[1,j]:8:3);
                                Write('  ]    [');
            For j := 1 to 3 Do  Write(XXmXX[2,j]:8:3);
                                Writeln('  ]');
            Write(XpXmXX[0,1]:8:3,XpXmXX[0,2]:8:3,'   [');
            for j := 1 to 3 Do  Write(XpXmXX[1,j]:8:3);
                                Write('  ]    [');
            For j := 1 to 3 Do  Write(XpXmXX[2,j]:8:3);
                                Writeln('  ]');
            Write(XXmXpX[0,1]:8:3,XXmXpX[0,2]:8:3,'   [');
            For j := 1 to 3 Do  Write(XXmXpX[1,j]:8:3);
                                Write('  ]    [');
            For j := 1 to 3 Do  Write(XXmXpX[2,j]:8:3);
                                Writeln('  ]');
            Write(XpXXmX[0,1]:8:3,XpXXmX[0,2]:8:3,'   [');
            For j := 1 to 3 Do  Write(XpXXmX[1,j]:8:3);
                                Write('  ]    [');
            For j := 1 to 3 Do  Write(XpXXmX[2,j]:8.3);
                                Writeln('  ]');
            Write(XmXXpX[0,1]:8:3,XmXXpX[0,2]:8:3,'   [');
```

```
            For j := 1 to 3 Do  Write(XmXXp[1,j]:8:3);
                                Write('  ]    [');
            For j := 1 to 3 Do  Write(XmXXp[2,j]:8:3);
                                Writeln('  ]');
            Writeln;
            Writeln;
            Flush1;
         End;
End;     { Procedure  Associative }



{}{}{}{}{}{}{}{}{}{}{}{}{}{}{}{}{}{}{}{}{}{}{}{}{}{}{}{}{}{}{}{}{}{}{}
{}{                                                                }{}
{}{ This procedure plots graphs in three dimensions. }{}
{}{                                                                }{}
{}{}{}{}{}{}{}{}{}{}{}{}{}{}{}{}{}{}{}{}{}{}{}{}{}{}{}{}{}{}{}{}{}{}{}

Procedure ShowGraph(Var  X,Y,Z : Mat;
                             p : Integer);

Var     GraphDriver,GraphMode,ErrorCode   : Integer;
        Theta,phi,r                       : Real;
        Zero,Hundred,q                    : Integer;
        st4                               : String[4];
        st6                               : String[6];


Procedure SetGraph;

Begin   { Procedure  SetGraph }
    GraphDriver := HercMono; { Hercules Monochrome = 7 }
    GraphMode := HercMonoHi; { 0 }
    InitGraph(GraphDriver,GraphMode,'');
    ErrorCode := GraphResult;
    If ErrorCode <> GrOK Then
       Begin { error }
          Writeln('Graphics error : ',GraphErrorMsg(ErrorCode));
          Writeln('Program aborted ...');
          Halt(1);
       End;  { error }
End;    { Procedure  SetGraph }


Procedure TransformCoordinates(Var Xinp,Yinp,Zinp  : Integer);

(* ****************************************************** *)
 *   This procedure rotates all axes.                     *)
(* ****************************************************** *)

```

```
Begin   { Procedure  TransformCoordinates }
     Xinp := Round(Xinp*Cos(Theta) + Yinp * Sin(Theta));
     Yinp := Round(Yinp*Cos(Theta) - Xinp * Sin(Theta));
End;    { Procedure  TransformCoordinates }


Procedure ProjectsToTwoDimensions(Var Xp,Yp : Integer;
                                          Zp : Integer );

(* ************************************************* *)
(*   This procedure projects three dimensions        *)
(*   into two dimensions.                            *)
(* ************************************************* *)

Var     Xp1,Yp1   : Real;

Begin   { Procedure  ProjectsToTwoDimensions }
     Xp1 := Yp - Xp*sin(Phi);
     Yp1 := Zp - Xp*Cos(Phi);
     Xp := Round(Xp1);
     Yp := Round(Yp1*3/4);
End;    { Procedure  ProjectsToTwoDimensions }


Procedure ThreeDimensions(Var Xinp,Yinp,Zinp : Integer);

(* ************************************************* *)
(*   This procedure projects three dimensions        *)
(*   into two dimensions and rotates coordinates.    *)
(* ************************************************* *)

Begin   { Procedure  ThreeDimensions }
     ProjectsToTwoDimensions(Xinp,Yinp,Zinp);
     TransformCoordinates(Xinp,Yinp,Zinp);
End;    { Procedure  ThreeDimensions }


Procedure DrawsArrow(Xp1,Yp1,Zp1,Xp2,Yp2,Zp2 : Integer);

(* ************************************************* *)
(*   This procedure draws arrow.                     *)
(* ************************************************* *)

Var     Theta1,L,Theta2 : Real;
        Sign            : Integer;

Begin   { Procedure  DrawsArrow }
     ThreeDimensions(Xp1,Yp1,Zp1);
     ThreeDimensions(Xp2,Yp2,Zp2);
     L := 6;
```

```
    Theta2 := Pi/8;
    If (Xp2 - Xp1) >= 0 Then Sign := 1
    Else If (Xp2 - Xp1) < 0 Then Sign := -1;
    If (Xp2 - Xp1) <> 0 Then  Theta1 := ArcTan((Yp2 - Yp1)/(Xp2 - Xp1))
    Else If (Yp2 - Yp1) >= 0 Then Theta1 := Pi/2
    Else Theta1 := -Pi/2;
    If ((Xp2 - Xp1) <> 0) Or ((Yp2 - Yp1) <> 0) Then
       Begin { check zero }
          Line(360 + Xp2,175 - Yp2,360 + Round
          (Xp2 - Sign*L*Cos(Theta1 - Theta2)),
          175 - Round(Yp2 - Sign*L*Sin(Theta1 - Theta2)));
          Line(360 + Xp2,175 - Yp2,
               360 + Round(Xp2 - Sign*L*Sin(Pi/2 - Theta1 - Theta2)),
               175 - Round(Yp2 - Sign*L*Cos(Pi/2 - Theta1 - Theta2)));
       End;  { check zero }
End;    { Procedure DrawsArrow }


Procedure DrawsSegment(Xp1,Yp1,Zp1,Xp2,Yp2,Zp2 : Integer);

(* ************************************************ *)
(*  This procedure draws a line when we give two    *)
(*  points.                                          *)
(* ************************************************ *)

Begin   { Procedure DrawsSegment }
    ThreeDimensions(Xp1,Yp1,Zp1);
    ThreeDimensions(Xp2,Yp2,Zp2);
    Line(360 + Xp1,175 - Yp1,360 + Xp2,175 - Yp2);
End;    { Procedure DrawsSegment }


Procedure DrawsVector(Xp1,Yp1,Zp1,Xp2,Yp2,Zp2 : Integer);

(* ************************************************ *)
(*  This procedure draws a vector (X,Y,Z), i.e.,    *)
(*  the vector starts from (0,0,0) to (X,Y,Z).      *)
(* ************************************************ *)

Begin   { Procedure DrawsVector }
    DrawsSegment(Xp1,Yp1,Zp1,Xp2,Yp2,Zp2);
    DrawsArrow(Xp1,Yp1,Zp1,Xp2,Yp2,Zp2);
End;    { Procedure DrawsVector }


Procedure DrawStar(X,Y,Z,scalar : Integer);

(* ************************************************ *)
(*  This procedure draws a star.                     *)
(* ************************************************ *)
```

```

Begin   { Procedure DrawStar }
    SetLineStyle(solidLn,0,NormWidth);
    DrawsSegment(X,Y,(Z-3),X,Y,(Z+3));
    DrawsSegment(X,(Y-3),Z,X,(Y+3),Z);
    DrawsSegment((X-3),Y,Z,(X+3),Y,Z);
    DrawsSegment(X,Y-3,Z+3,X,Y+3,Z-3);
    DrawsSegment(X-3,Y+3,Z,X+3,Y-3,Z);
    DrawsSegment(X+3,Y,Z-3,X-3,Y,Z+3);
End;    { Procedure DrawStar }


Procedure DrawText(Xinp,Yinp,Zinp : Integer;
                             lett : String ;
                              X,Y : Integer);

(* ************************************************* *)
(*   This procedure writes letters into graphic      *)
(*   mode.                                           *)
(* ************************************************* *)

Begin   { Procedure DrawText }
    ThreeDimensions(Xinp,Yinp,Zinp);
    OutTextXY((360+Xinp+X),(175-Yinp-Y),lett);
End;    { Procedure  DrawText }


Procedure DrawScal(q : Integer);

(* ************************************************* *)
(*   This procedure draws scales on three axes.      *)
(* ************************************************* *)

Var     Ci,no,l : Integer;
        w       : Real;

Begin   { Procedure  DrawScal }
    If  q = 1  Then
       Begin
          ci := 16;
          no := 21;
       End
    Else
       Begin
          ci := 20;
          no := 17;
       End;
    w := 2/q;
    SetLineStyle(solidLn,0,NormWidth);
       For l := 0 to no Do
```

```
            Begin
               DrawsSegment((-160+(l*ci)),0,-2,(-160+(l*ci)),0,2);
               DrawsSegment(0,(-160+(l*ci)),-2,0,(-160+(l*ci)),2);
               DrawsSegment(0,-2,(-160+(l*ci)),0,2,(-160+(l*ci)));
            End;
      SetLineStyle(solidLn,0,ThickWidth);
         For l := 0 to 4 Do
            Begin
               If l <> 2   then
                  begin
                     DrawsSegment((-160+(l*80)),0,-2,(-160+(l*80)),0,2);
                     DrawsSegment(0,(-160+(l*80)),-2,0,(-160+(l*80)),2);
                     DrawsSegment(0,-2,(-160+(l*80)),0,2,(-160+(l*80)));
                  End
               Else
                  End;
      str(w:3:1,st4);
      DrawText(160,0,0,st4,-25,-2);             { scal x }
      DrawText(0,160,0,st4,-7,-5);              { scal y }
      DrawText(0,0,160,st4,-30,2);              { scal z }
      DrawText(-160,0,0,'-'+st4,1,9);             { scal -x }
      DrawText(0,-160,0,'-'+st4,-15,-5);          { scal -y }
      DrawText(0,0,-160,'-'+st4,10,2);            { scal -z }
 End;    { Procedure DrawScal }


 Procedure ProjectVector(X,Y,Z,Width,check : Integer);

 (* ************************************************ *)
 (*  This procedure projects a vector on three       *)
 (*  planes.                                          *)
 (* ************************************************ *)

 Begin   { Procedure ProjectVector }
      SetLineStyle(3,0,Width);
      DrawsVector(0,0,0,0,Y,Z);
      DrawsVector(0,0,0,X,0,Z);
      DrawsVector(0,0,0,X,Y,0);
      Case check of
           1 : SetLineStyle(solidLn,0,NormWidth);
           2 : SetLineStyle(1,0,NormWidth);
      End;
      DrawsSegment(X,Y,Z,0,Y,Z);
      DrawsSegment(X,Y,Z,X,0,Z);
      DrawsSegment(X,Y,Z,X,Y,0);
      DrawsSegment(X,0,0,X,Y,0);
      DrawsSegment(X,0,0,X,0,Z);
      DrawsSegment(0,Y,0,X,Y,0);
      DrawsSegment(0,Y,0,0,Y,Z);
      DrawsSegment(0,0,Z,0,Y,Z);
```

```
    DrawsSegment(0,0,Z,X,0,Z);
End;    { Procedure ProjectVector }


Procedure DrawLineVectorScalar(Var X                     : Mat;
                                   check,linetype,width : Integer;
                               Var q                     : Integer);

(* ********************************************** *)
(*   This procedure projects a vector on three      *)
(*   planes, and projects a scalar on an axis.      *)
(* ********************************************** *)

Var     Zero,ComponentX,ComponentY,ComponentZ,scalar : Integer;

Begin   { Procedure  DrawLineVectorScalar }
    Zero := 0;
    If (P >= 6) And (P <= 15) Then  q := 5
    Else   q := 1;
{    q := 5;}
    Scalar := trunc(X[0,check] * q * 80);
    ComponentX := trunc(X[check,1] * q * 80);
    ComponentY := trunc(X[check,2] * q * 80);
    ComponentZ := trunc(X[check,3] * q * 80);
    SetLineStyle(0,4,Width);
    DrawsVector(Zero,Zero,Zero,ComponentX,ComponentY,ComponentZ);
    Case check of
        1 : DrawStar(Zero,scalar,Zero,scalar);
        2 : DrawStar(Zero,Zero,scalar,scalar);
    End;
    ProjectVector(ComponentX,ComponentY,componentZ,Width,check);
End;    { Procedure DrawLineVectorScalar }


Begin   { Procedure ShowGraph }
    theta := 0;
    phi := -0.9;
    SetGraph;

(* ************  DRAW  X,Y,and Z axes   ************ *)
    SetLineStyle(0,1,1);
    DrawsSegment(-200,0,0,200,0,0);
    DrawsSegment(0,-210,0,0,210,0);
    DrawsSegment(0,0,-170,0,0,185);
(* ********************************************** *)

(* *************    DRAW  XY Plane      ************ *)
    SetLineStyle(0,0,2);
    DrawsSegment(-100,-160,0,-100,-80,0);
    DrawsSegment(-100,-160,0,100,-160,0);
```

```
    DrawsSegment(100,160,0,0,160,0);
    DrawsSegment(100,-160,0,100,160,0);
(* ************************************************** *)

(* *************     DRAW  YZ Plane      ************ *)
    SetLineStyle(0,0,2);
    DrawsSegment(0,-160,-100,0,-160,100);
    DrawsSegment(0,-160,-100,0,0,-100);
    DrawsSegment(0,-160,100,0,160,100);
    DrawsSegment(0,80,-100,0,160,-100);
    DrawsSegment(0,160,0,0,160,100);
    DrawsSegment(0,160,-66,0,160,-100);
(* ************************************************** *)

(* *************     Draw  XZ  Plane     ************ *)
    SetLineStyle(SolidLn,0,ThickWidth);
    DrawsSegment(-100,0,38,-100,0,100);
    DrawsSegment(-100,0,100,100,0,100);
    DrawsSegment(0,0,-100,100,0,-100);
    DrawsSegment(100,0,-100,100,0,100);
(* ************************************************** *)

    DrawLineVectorScalar(Z,1,SolidLn,ThickWidth,q);
    DrawLineVectorScalar(Z,2,SolidLn,NormWidth,q);
    DrawScal(q);
    SetTextJustify(CenterText,CenterText);
    SetTextStyle(GothicFont,HorizDir,1);
    str(p,st4);
    OutTextXY(350,20,'FELLOURIS PRODUCT');
    OutTextXY(310,35,'D['+st4+']');
    OutTextXY(350,320,'(J)');
    DrawText(200,0,0,'X',-10,-10);
    DrawText(0,210,0,'Y',10,0);
    DrawText(0,0,185,'Z',0,5);
    Repeat Until ReadKey = #13;
    CloseGraph;
End;    { Procedure ShowGraph }

{}{}{}{}{}{}{}{}{}{}{}{}{}{}{}{}{}{}{}{}{}{}{}{}{}{}{}{}{}{}{}{}
{}{                                                  }{}
{}{      Finish plotting graph in three dimensions.  }{}
{}{                                                  }{}
{}{}{}{}{}{}{}{}{}{}{}{}{}{}{}{}{}{}{}{}{}{}{}{}{}{}{}{}{}{}{}{}



Begin   { Main Program }
    Clrscr;
    GetData(dataX,dataY,n);
    For i := 1 to n Do
```

```
{              begin
                  ChangeVecMatToMat(dataX,X,i);
                  ChangeVecMatToMat(dataY,Y,i);
                  Zorn(X,Y,Z);
{                  Calculation(X,Y,Z);}
{                  CheckFlexible(X,Y,i);
                  Associative(X,3); } { or Associative(X,4) for 4th power }
                  ChangeMatToVecMat(Z,dataZ,i);
                      if (n - i) > 0 then
                             Begin
                                 Writeln('It has about ',n-i+1,' pictures of graphic');
                                 Writeln;
                                 Writeln('Would you like to look it  [Y/N] ? ');
                                 Repeat
                                 ans := Upcase(readkey);
                                 Until (ans = 'N') Or (ans = 'Y');
                                 If (ans = 'N') Then Halt;
                             End
                       Else   Writeln('It has about 1 picture of graphic');
                  ShowGraph(X,Y,Z,i);
              End;
{    writeln('WAIT FOR A MOMENT...');
      KeepData(dataX,dataY,dataZ,n);}
      Writeln('O.K.  FINISH..!      PLEASE PRESS RETURN');
      Readln;
End.     { Main Program }
```

APPENDIX B

COMPUTATIONAL RESULTS

These data are for the products of the Zorn vector matrices.

A realization of the split octonion algebra is via the Zorn vector matrices

$$\begin{bmatrix} a & X^{\sim} \\ Y^{\sim} & b \end{bmatrix},$$

where a and b are scalars and X~ and Y~ are 3-vectors, with the product defined as

$$\begin{bmatrix} a & X^{\sim} \\ Y^{\sim} & b \end{bmatrix} \begin{bmatrix} c & U^{\sim} \\ V^{\sim} & d \end{bmatrix} = \begin{bmatrix} ac - X^{\sim}V^{\sim} & aU^{\sim} + dX^{\sim} + [Y^{\sim} \times V^{\sim}] \\ cY^{\sim} + bV^{\sim} + [X^{\sim} \times U^{\sim}] & Y^{\sim}U^{\sim} - bd \end{bmatrix}$$

x denotes the usual vector product,

and

$$\begin{bmatrix} e & W^{\sim} \\ Z^{\sim} & f \end{bmatrix} = \begin{bmatrix} ac - X^{\sim}V^{\sim} & aU^{\sim} + dX^{\sim} + [Y^{\sim} \times V^{\sim}] \\ cY^{\sim} + bV^{\sim} + [X^{\sim} \times U^{\sim}] & Y^{\sim}U^{\sim} - bd \end{bmatrix}$$

The following table gives the values of scalars [a,b,c,d,e,f] and vectors [X~,Y~,U~,V~,W~,Z~] :

N	a / c / e	b / d / f	X~ / U~ / W~			Y~ / V~ / Z~		
1	1.000	1.000	0.500	0.707	0.500	-0.500	-0.500	-0.707
	1.000	1.000	0.333	0.667	0.667	-0.707	-0.500	-0.500
	1.957	1.972	0.729	1.624	1.064	-1.069	-1.167	-1.109
			$\lvert W^{\sim} \rvert$ =		2.074	$\lvert Z^{\sim} \rvert$ =		1.932
2	1.000	1.000	0.333	0.667	0.667	-0.667	-0.333	-0.667
	1.000	1.000	0.500	0.750	0.433	-0.500	-0.433	-0.750
	1.956	1.872	0.794	1.250	1.222	-1.378	-0.577	-1.501
			$\lvert W^{\sim} \rvert$ =		1.920	$\lvert Z^{\sim} \rvert$ =		2.118
3	1.000	1.000	0.750	0.433	0.500	-0.250	-0.750	-0.612
	1.000	1.000	0.600	0.400	0.693	-0.600	-0.600	-0.529
	1.974	1.874	1.380	1.068	0.893	-0.750	-1.570	-1.101
			$\lvert W^{\sim} \rvert$ =		1.960	$\lvert Z^{\sim} \rvert$ =		2.059
4	1.000	1.000	0.620	0.380	0.646	-0.500	-0.667	-0.500
	1.000	1.000	0.348	0.780	0.520	-0.570	-0.660	-0.490
	1.921	1.954	0.965	1.200	1.116	-1.376	-1.425	-0.639
			$\lvert W^{\sim} \rvert$ =		1.902	$\lvert Z^{\sim} \rvert$ =		2.081
5	1.000	1.000	0.614	0.463	0.639	-0.577	-0.430	-0.694
	1.000	1.000	0.624	0.500	0.600	-0.710	-0.515	-0.480
	1.981	1.991	1.087	1.179	1.231	-1.329	-0.915	-1.156
			$\lvert W^{\sim} \rvert$ =		2.021	$\lvert Z^{\sim} \rvert$ =		1.984
6	1.000	1.000	0.600	0.600	0.529	-0.540	-0.620	-0.569
	1.000	1.000	-0.630	-0.712	-0.310	0.570	0.600	0.560
	0.002	0.042	-0.036	-0.134	0.248	0.221	-0.167	-0.058
			$\lvert W^{\sim} \rvert$ =		0.284	$\lvert Z^{\sim} \rvert$ =		0.283

7	1.000	1.000	[	0.575	0.424	0.700	]	[	-0.607	-0.611	-0.508	]
	1.000	1.000	[	-0.520	-0.580	-0.627	]	[	0.670	0.600	0.437	]
	0.054	0.011	[	0.093	-0.231	0.118	]	[	0.203	-0.014	-0.184	]
				| W~| =		0.276			| Z~| =		0.274	
8	1.000	1.000	[	0.410	0.420	0.810	]	[	-0.660	-0.630	-0.409	]
	1.000	1.000	[	-0.460	-0.350	-0.769	]	[	0.493	0.610	0.620	]
	0.039	0.161	[	-0.191	0.278	-0.051	]	[	-0.206	-0.077	0.261	]
				| W~| =		0.341			| Z~| =		0.341	
9	1.000	1.000	[	0.575	0.600	0.556	]	[	-0.630	-0.531	-0.567	]
	1.000	1.000	[	-0.580	-0.500	-0.643	]	[	0.533	0.567	0.628	]
	0.004	0.005	[	-0.017	0.193	-0.161	]	[	-0.205	0.083	0.122	]
				| W~| =		0.252			| Z~| =		0.252	
10	1.000	1.000	[	0.720	0.420	0.552	]	[	-0.532	-0.559	-0.636	]
	1.000	1.000	[	-0.600	-0.693	-0.400	]	[	0.500	0.550	0.669	]
	0.040	0.039	[	0.096	-0.235	0.139	]	[	0.183	-0.052	-0.214	]
				| W~| =		0.289			| Z~| =		0.286	
11	1.000	1.000	[	-0.550	-0.500	-0.669	]	[	0.660	0.610	0.439	]
	1.000	1.000	[	0.860	0.110	0.498	]	[	-0.730	-0.300	-0.614	]
	0.038	0.147	[	0.067	-0.305	0.076	]	[	-0.245	0.009	0.194	]
				| W~| =		0.322			| Z~| =		0.313	
12	1.000	1.000	[	-0.560	-0.600	-0.570	]	[	0.540	0.500	0.662	]
	1.000	1.000	[	0.630	0.570	0.527	]	[	-0.691	-0.480	-0.540	]
	0.017	0.026	[	0.118	-0.196	0.043	]	[	-0.142	-0.044	0.181	]
				| W~| =		0.233			| Z~| =		0.234	
13	1.000	1.000	[	-0.500	-0.707	-0.500	]	[	0.500	0.500	0.707	]
	1.000	1.000	[	0.333	0.667	0.667	]	[	-0.500	-0.707	-0.500	]
	0.000	0.028	[	0.083	-0.143	0.063	]	[	-0.138	-0.040	0.109	]
				| W~| =		0.177			| Z~| =		0.180	
14	1.000	1.000	[	-0.330	-0.767	-0.567	]	[	0.667	0.303	0.697	]
	1.000	1.000	[	0.500	0.751	0.433	]	[	-0.500	-0.433	-0.750	]
	0.078	0.137	[	0.245	0.136	-0.271	]	[	0.261	-0.271	0.083	]
				| W~| =		0.390			| Z~| =		0.385	
15	1.000	1.000	[	-0.750	-0.433	-0.500	]	[	0.750	0.612	0.250	]
	1.000	1.000	[	0.693	0.400	0.600	]	[	-0.600	-0.600	-0.529	]
	0.026	0.085	[	-0.231	0.214	0.017	]	[	0.090	0.115	-0.279	]
				| W~| =		0.315			| Z~| =		0.315	
16	1.000	1.000	[	-0.600	-0.540	-0.590	]	[	0.500	0.766	0.400	]
	1.000	1.000	[	-0.570	-0.530	-0.628	]	[	0.550	0.200	0.721	]
	1.863	1.942	[	-0.698	-1.210	-1.539	]	[	1.076	0.925	1.131	]
				| W~| =		2.079			| Z~| =		1.815	
17	1.000	1.000	[	-0.424	-0.700	-0.575	]	[	0.811	0.508	0.291	]
	1.000	1.000	[	-0.800	-0.500	-0.332	]	[	0.900	0.387	0.200	]
	1.767	1.999	[	-1.235	-1.100	-1.050	]	[	1.656	1.214	0.143	]
				| W~| =		1.959			| Z~| =		2.058	
18	1.000	1.000	[	-0.765	-0.321	-0.559	]	[	0.141	0.700	0.700	]
	1.000	1.000	[	-0.433	-0.750	-0.500	]	[	0.328	0.900	0.287	]
	1.700	1.936	[	-1.627	-0.882	-1.162	]	[	0.210	1.460	1.422	]
				| W~| =		2.185			| Z~| =		2.048	
19	1.000	1.000	[	-0.265	-0.916	-0.300	]	[	0.580	0.610	0.540	]
	1.000	1.000	[	-0.710	-0.498	-0.498	]	[	0.300	0.730	0.614	]
	1.932	1.984	[	-0.995	-1.608	-0.558	]	[	1.187	1.421	0.636	]
				| W~| =		1.971			| Z~| =		1.957	
20	1.000	1.000	[	-0.440	-0.690	-0.580	]	[	0.520	0.299	0.800	]
	1.000	1.000	[	-0.610	-0.700	-0.130	]	[	0.612	0.210	0.783	]
	1.868	1.630	[	-0.984	-1.308	-0.784	]	[	0.816	0.806	1.470	]
				| W~| =		1.814			| Z~| =		1.864	